LEARNING MATERIAL
ON
CO-ORDINATE GEOMETRY-2D

SEMESTER : I
DEPARTMENT : MATHEMATICS AND SCIENCE
SUBJECT NAME : ENGINEERING MATHEMATICS-I
SUBJECT CODE : TH.3

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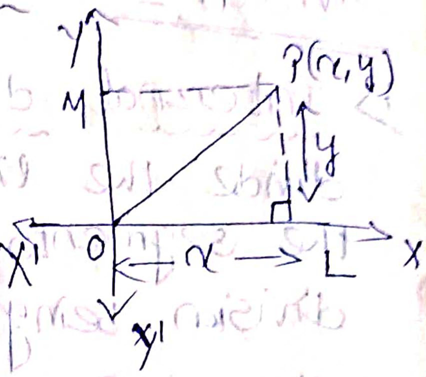
CO-ORDINATE GEOMETRY

Cartesian co-ordinate of a Point :-

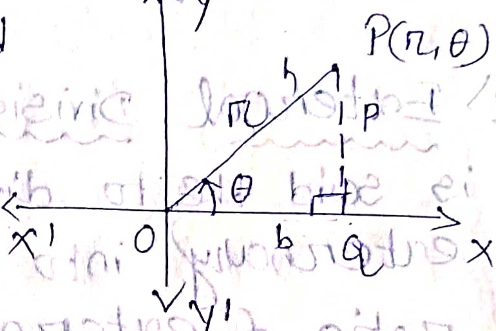


Distance from x-axis : y

Distance from y-axis : x

Distance from origin : $\sqrt{x^2+y^2}$

Polar co-ordinate system :-



r = Distance between P and origin.

θ = Angle between x-axis and P i.e inclination.

Relation between cartesian and Polar system of co-ordinate :-

In $\triangle POQ$, $\cos\theta = \frac{b}{h} = \frac{OQ}{OP} = \frac{x}{r}$

$$\Rightarrow \boxed{x = r\cos\theta}$$

$\sin\theta = \frac{p}{h} = \frac{PQ}{OP} = \frac{y}{r} \Rightarrow \boxed{y = r\sin\theta}$

Thus, $x^2+y^2 = r^2\cos^2\theta + r^2\sin^2\theta$

$$\Rightarrow x^2+y^2 = r^2(\cos^2\theta+\sin^2\theta) = r^2$$

$$\Rightarrow \boxed{r = \sqrt{x^2+y^2}}$$

Further, $\tan\theta = \frac{p}{b} = \frac{PQ}{OQ} = \frac{y}{x}$

$$\Rightarrow \boxed{\theta = \tan^{-1}(y/x)}$$

Hence, $P(x,y) \equiv P(r\cos\theta, r\sin\theta)$, where

$r = \sqrt{x^2+y^2}$ be the distance between the point and origin.

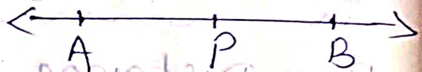
$\theta = \tan^{-1}(y/x)$ be the angle between P and x-axis.

STRAIGHT LINES

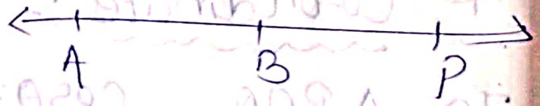
Division :-

i) <u>Internal division</u> :- If $A-P-B$, then P is said to divide the line segment $\overline{AB}$ internally into the segments $\overline{PA}$ and $\overline{PB}$; the ratio of internal division being given by $PA : PB$ or $PB : PA$.

Here $PA + PB = AB$.



ii) <u>External Division</u> :- If $P-A-B$ or $A-B-P$, then P is said ~~the~~ to divide the line segment $\overline{AB}$ externally into the segments $\overline{PA}$ and $\overline{PB}$; the ratio of external division is given by $PA : PB$ or $PB : PA$.



Here $|PA - PB| = AB$

N.B:- If P divides $\overline{AB}$ or the line segment joining A and B in ratio $m:n$, then

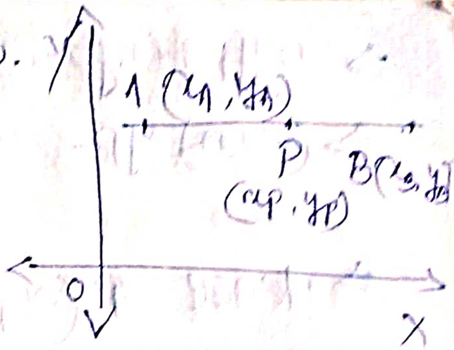
$$\boxed{\frac{PA}{PB} = \frac{m}{n}}$$

Internal Division Formula :-

If $P(x_P, y_P)$ divides $\overline{AB}$, the line segment joining $A(x_A, y_A)$ and $B(x_B, y_B)$ internally so that $\frac{PA}{PB} = \frac{m}{n}$ then

$$x_P = \frac{m x_B + n x_A}{m+n} \text{ and } y_P = \frac{m y_B + n y_A}{m+n}$$

Proof:- Case-(a) If $\overline{AB} \parallel$ to x-axis.



then $PA = |x_P - x_A|$ and
$PB = |x_B - x_P|$

Now $\frac{PA}{PB} = \frac{m}{n} \Rightarrow \frac{|x_P - x_A|}{|x_B - x_P|} = \frac{m}{n}$

$\Rightarrow \frac{x_P - x_A}{x_B - x_P} = \frac{m}{n}$

$\Rightarrow n x_P - n x_A = m x_B - m x_P$

$\Rightarrow (m+n) x_P = m x_B + n x_A$

$\Rightarrow x_P = \frac{m x_B + n x_A}{m+n}$

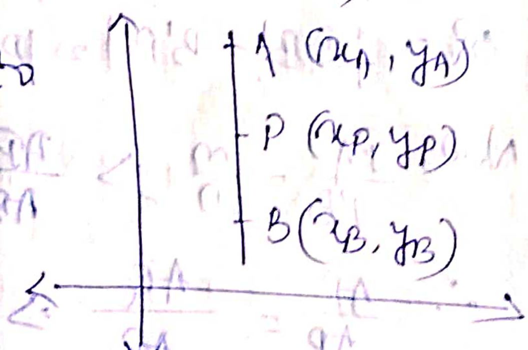
In this case, $y_A = y_B = y_P$

$\Rightarrow m y_B + n y_A = m y_P + n y_P = (m+n) y_P$

$\Rightarrow y_P = \frac{m y_B + n y_A}{m+n}$

Hence, $P(x_P, y_P) = P\left(\frac{m x_B + n x_A}{m+n}, \frac{m y_B + n y_A}{m+n}\right)$

Case-b:- If $\overline{AB}$ is parallel to y-axis, then



$x_A = x_B = x_P$

So, $m x_B + n x_A = m x_P + n x_P$

$\Rightarrow m x_B + n x_A = (m+n) x_P$

$\Rightarrow x_P = \frac{m x_B + n x_A}{m+n}$

Again, $PA = |y_A - y_P|$ and $PB = |y_P - y_B|$

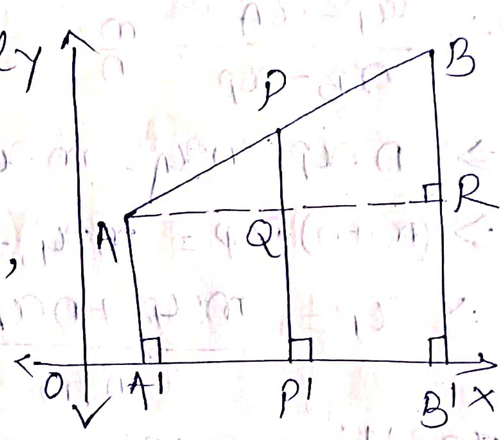
$\frac{PA}{PB} = \frac{m}{n} \Rightarrow \frac{|y_A - y_P|}{|y_P - y_B|} = \frac{m}{n} \Rightarrow \frac{y_A - y_P}{y_P - y_B} = \frac{m}{n}$

$\Rightarrow m y_A - n y_P = m y_P - m y_B$

$\Rightarrow m y_P + n y_P = m y_B + n y_A$

$\Rightarrow (m+n) y_P = \dfrac{m y_B + n y_A}{m+n}$

Hence $P(x_P, y_P) = P\left(\dfrac{m x_B + n x_A}{m+n}, \dfrac{m y_B + n y_A}{m+n}\right)$



Case-c :- if $\overline{AB}$ is not parallel to any of the axes.

Let $\overline{AA'}$, $\overline{PP'}$ and $\overline{BB'}$ be $\perp$ to x-axis, meeting it at A', P' and B' respectively.

Again, let $\overline{AR}$ $\perp$ $\overline{BB'}$.

Now, ΔPAQ and ΔBAR are similar.

$$\Rightarrow \frac{AP}{AB} = \frac{AQ}{AR} = \frac{PQ}{BR}$$

$AQ = A'P' = |OP' - OA'| = |x_P - x_A|$

$AR = A'B' = |OB' - OA'| = |x_B - x_A|$

$PQ = |PP' - QP'| = |PP' - AA'| = |y_P - y_A|$

$BR = |BB' - B'R| = |BB' - AA'| = |y_B - y_A|$

Now, $\dfrac{PA}{PB} = \dfrac{m}{n} \Rightarrow \dfrac{AP}{AB} = \dfrac{m}{m+n}$

$\therefore \dfrac{AP}{AB} = \dfrac{AQ}{AR} \Rightarrow \dfrac{m}{m+n} = \dfrac{|x_P - x_A|}{|x_B - x_A|} = \dfrac{x_P - x_A}{x_B - x_A}$

$\Rightarrow m(x_B - x_A) = (m+n)(x_P - x_A)$

$\Rightarrow m x_B - m x_A = m x_P - m x_A + n x_P - n x_A$

$\Rightarrow m x_B + n x_A = (m+n) x_P$

$\Rightarrow x_P = \dfrac{m x_B + n x_A}{m+n}$

Similarly, $\dfrac{AP}{AB} = \dfrac{PQ}{BR} \Rightarrow \dfrac{m}{m+n} = \dfrac{|y_P - y_A|}{|y_B - y_A|}$

$$\Rightarrow \frac{m}{m+n} = \frac{y_P - y_A}{y_B - y_A}$$

$$\Rightarrow m(y_B - y_A) = (m+n)(y_P - y_A)$$

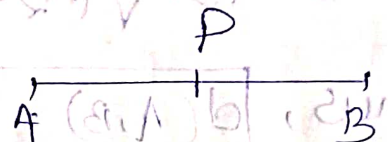
$$\Rightarrow my_B - my_A = my_P + ny_P - my_A - ny_A$$

$$\Rightarrow my_B + ny_A = (m+n)\,y_P$$

$$\Rightarrow y_P = \frac{my_B + ny_A}{m+n}$$

Hence, $\boxed{P(x_P, y_P) = P\left(\frac{mx_B + nx_A}{m+n}, \frac{my_B + ny_A}{m+n}\right)}$

Mid-Point Formula :-



If $P(x, y)$ is the mid point of the line segment joining $A(x_1, y_1)$ and $B(x_2, y_2)$, then

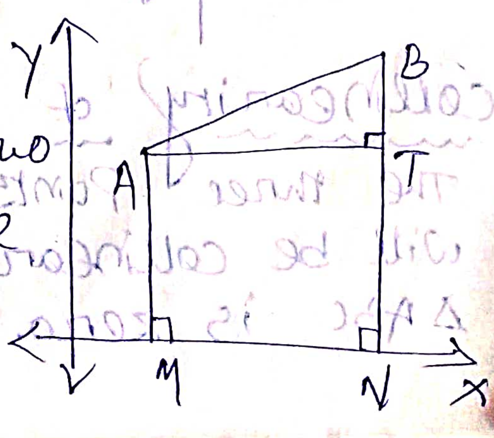
$$\boxed{x = \left(\frac{x_1 + x_2}{2}\right) \text{ and } y = \left(\frac{y_1 + y_2}{2}\right)}$$

External Division Formula :-

If $P(x_P, y_P)$ divides $\overline{AB}$, the line segment joining $A(x_A, y_A)$ and $B(x_B, y_B)$ externally so that $\frac{PA}{PB} = \frac{m}{n}$, then

$$\boxed{x_P = \frac{mx_B - nx_A}{m-n} \text{ and } y_P = \frac{my_B - ny_A}{m-n}}$$

Distance Formula :-



If $A(x_1, y_1)$ and $B(x_2, y_2)$ be two points in the plane then the distance $|AB|$ or $d(A, B)$ is given by -

$$d(A,B) = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

Proof:- suppose $\overline{AB}$ is not parallel to either of the axes.

Let $\overline{AM}$ and $\overline{BN}$ be the perpendicular drawn from A and B to x-axis. and $\overline{AT}$ be the $\perp$ from A to $\overline{BN}$.

In ΔABT, $AB^2 = AT^2 + BT^2 = MN^2 + BT^2$

$$\Rightarrow AB^2 = (x_2-x_1)^2+(y_2-y_1)^2$$

$$\Rightarrow AB = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

Thus, $\boxed{d(A,B) = AB = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}}$

Corollary:- The distance of a point $P(x,y)$ from the origin is $OP = \sqrt{x^2+y^2}$.

Formula for Area of a triangle:-

The area of a triangle with vertices at $A(x_1,y_1)$, $B(x_2,y_2)$ and $C(x_3,y_3)$ is $|\Delta|$, where

$$\Delta = \frac{1}{2}\left[x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)\right]$$

$$= \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

Collinearity of Three Points:

The three points $A(x_1,y_1)$, $B(x_2,y_2)$ and $C(x_3,y_3)$ will be collinear iff the area of the triangle ΔABC is zero.

i.e $\left[x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)\right]=0$

$$\Rightarrow \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = 0.$$

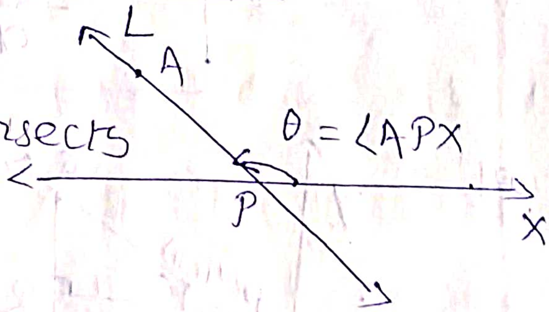
Inclination and slope of a line :-

Inclination :-

Inclination of a line is a real number 'θ' as defined below -

1. If a line L is Parallel to x-axis or coincides with x-axis, then $\boxed{\theta = 0.}$
2. If L is not parallel to x-axis, then let it intersects x-axis at P.

Then $\theta = m\angle APX$.



3. Inclination of $\overline{AB}$, $\overrightarrow{AB}$ or $\overrightarrow{BA}$ is defined as inclination of $\overleftrightarrow{AB}$.

Note :-

* If θ is the inclination of a line, then $\boxed{0 \le \theta < \pi}$
* Parallel lines have the same inclination and conversely.
* Inclination of Perpendicular lines differ by $\pi/2$.
* Angle between the x-axis and line in anticlockwise direction $= \theta$

Slope (Gradient) of a Non-vertical line :-

The slope of a non-vertical line is given by

$$\boxed{m = \tan\theta}$$

where θ = The inclination of the line.

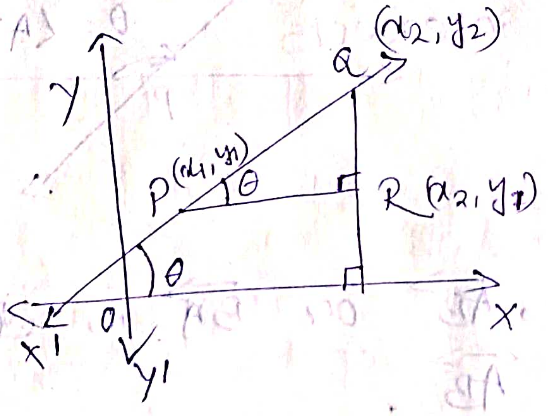
Inclination $(\theta) \in [0, \pi) \Rightarrow$ slope $(m) \in \mathbb{R}$

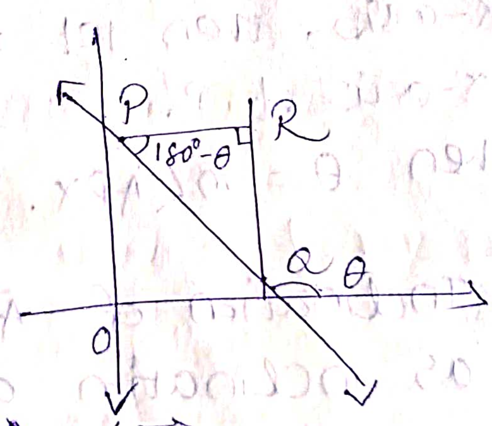
- Slope of a vertical line is not defined as $\tan(\frac{\pi}{2})$ is not defined.

Theorem:- If $P(x_1, y_1)$ and $Q(x_2, y_2)$ are two distinct points on a non-vertical line, then slope of the line $\overleftrightarrow{PQ}$ is given by

$$\boxed{m = \frac{y_2 - y_1}{x_2 - x_1}}$$

Proof:-





Let $\overleftrightarrow{QR} \perp$ X-axis and $\overrightarrow{PR} \perp \overleftrightarrow{QR}$

Let m be the slope of $\overrightarrow{PQ}$, having inclination θ.

Then,

i) For $\theta < \pi/2$, $m = \tan\theta = \frac{QR}{PR} = \frac{|y_2 - y_1|}{|x_2 - x_1|}$

ii) For $\theta > \frac{\pi}{2}$, $m = \tan\theta = -\tan(180° - \theta) = -\frac{QR}{PR}$

$$\Rightarrow m = -\frac{|y_2 - y_1|}{|x_2 - x_1|}$$

Hence $\boxed{m = \frac{y_2 - y_1}{x_2 - x_1}}$

Note:-

* $m > 0 \Rightarrow \theta < \frac{\pi}{2}$ and $m < 0 \Rightarrow \theta > \frac{\pi}{2}$

  $m = 0 \Rightarrow \theta = 0^\circ$

Theorem:-

If a Pair of lines L_1 and L_2 have slopes m_1 and m_2 respectively, then

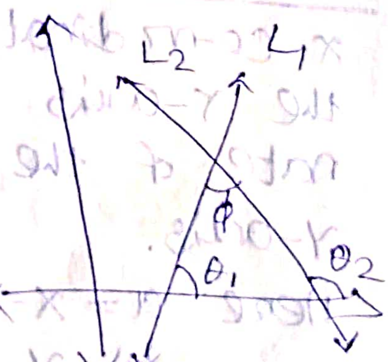
i) $m_1 = m_2 \Leftrightarrow L_1 \parallel L_2$ i.e L_1 Parallel to L_2 or
$L_1 = L_2$ i.e L_1 coincides with L_2

ii) $m_1 \cdot m_2 = -1 \Leftrightarrow L_1 \perp L_2$ i.e L_1 is Perpendicular to L_2.

Angle Between a Pair of intersecting lines:-

If ϕ measures an angle between the intersecting lines L_1 and L_2 with slopes m_1 and m_2 respectively then

$$\boxed{\tan\phi = \pm \frac{m_1 - m_2}{1 + m_1 \cdot m_2}}$$



Note:-

* If $\frac{m}{n} > 0 \Rightarrow$ Division is internal.

  If $\frac{m}{n} < 0 \Rightarrow$ Division is External.

Locus and its Equation :-

Locus :- A set of Points satisfying certain condition or conditions is called a locus.

A Point belonging to the Locus is called a "Point on the Locus".

Chord of a Locus :- A line-segment joining two Points on a locus is a chord of the locus.

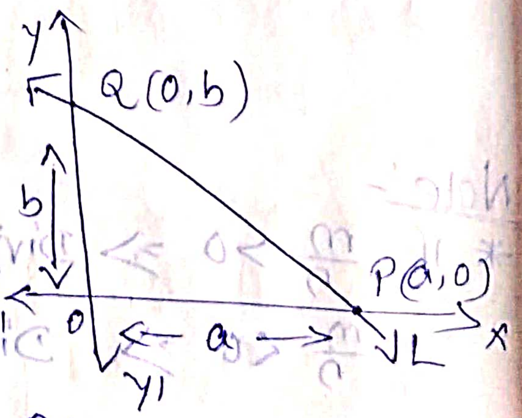
Equation of a Locus :- The equation satisfied by the co-ordinates of all the Points on a Locus and by no others, is called the equation of the locus.

Equation of a Straight Line :-

① **Slope-Intercept Form** :-

Intercept :- The x-intercept of a line is the x-co-ordinate of the point where it intersects the x-axis and its y-intercept is the y-co-ordinate of the Point where it intersects the y-axis.

Here a = x-intercept of L.
b = y-intercept of L.



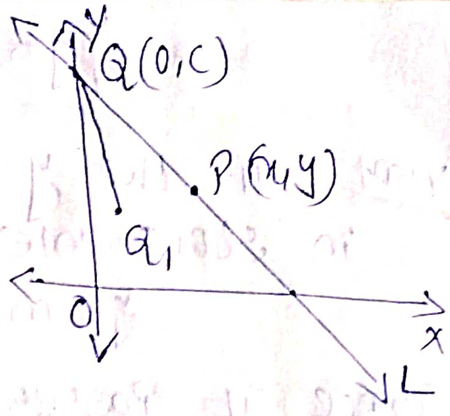
A line Parallel to x-axis has no x-intercept.

A line Parallel to y-axis has no y-intercept.

Theorem :- If a line has slope 'm' and y-intercept 'c' then its equation is-

$$\boxed{y = mx + c}$$

Proof.- Let $P(x,y)$ be any point on L, other than Q.



slope of line $(m) = \frac{y-c}{x-0} = \frac{y-c}{x}$

$\Rightarrow m = \frac{y-c}{x}$

$\Rightarrow mx = y-c \Rightarrow \boxed{y = mx+c}$

Thus, the co-ordinates of all the points on L satisfy the equation $y = mx+c$.

Suppose $Q_1 \notin L$ and $Q_1(x_1, y_1)$ be the co-ordinate.

and $y_1 = mx_1 + c$.

we have to show $x_1 \neq 0$.

If $x_1 = 0 \Rightarrow y_1 = c$

which means $Q_1(x_1, y_1) = Q_1(0,c)$ coincides with Q.

$\Rightarrow Q_1 \in L$, which is contrary to our assumption.

Now $y_1 = mx_1 + c \Rightarrow m = \frac{y_1 - c}{x_1} = \frac{y_1 - c}{x_1 - 0}$

$\Rightarrow m =$ slope of $\overleftrightarrow{PQ}$ = slope of $\overleftrightarrow{QQ_1}$

$\Rightarrow$ since $\overleftrightarrow{PQ}$ and $\overleftrightarrow{QQ_1}$ have the common point Q, it is clear that both lines are coincides.

$\Rightarrow Q_1(x_1, y_1) \in L$, which is a contradiction.

Hence, by definition of equation to a locus it is proved that $y = mx+c$ is the equation of the given line.

② Slope-Point Form:-

Let a line have slope 'm' and let it pass through a point $Q(x_1, y_1)$. Then its equation is given by-

$$\boxed{y - y_1 = m(x - x_1)}$$

Proof:- If the y-intercept of the line is c, then in slope-intercept form, its equation be-

$$y = mx + c. \quad \text{—— (1)}$$

Since it passes through (x_1, y_1), we have-

$$y_1 = mx_1 + c. \quad \text{—— (2)}$$

From eqn (1) and (2), we have-

$$y - y_1 = mx + c - mx_1 - c$$

$$\Rightarrow \boxed{y - y_1 = m(x - x_1)}$$

(3) Two-Point Form :-

Let a line pass through two given points $Q(x_1, y_1)$ and $R(x_2, y_2)$. Then the equation of the line is given by-

$$\boxed{y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)}$$

Proof:- Since the line passes through $Q(x_1, y_1)$ and $R(x_2, y_2)$, its slope is-

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Also the line passes through the point (x_1, y_1) and has slope m. So its equation be-

$$y - y_1 = m(x - x_1) \quad \text{—— (1)}$$

$$\Rightarrow \boxed{y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)}$$

④ **Intercept Form:-**

Let a line have x-intercept a and y-intercept b, then the equation is-

$$\boxed{\frac{x}{a} + y/b = 1}$$

Proof:- The line has x-intercept a and y-intercept b. so it passes through (a,0) and (0,b) and hence, the eqn. of line-

$$(y-0) = \left(\frac{b-0}{0-a}\right)(x-a)$$

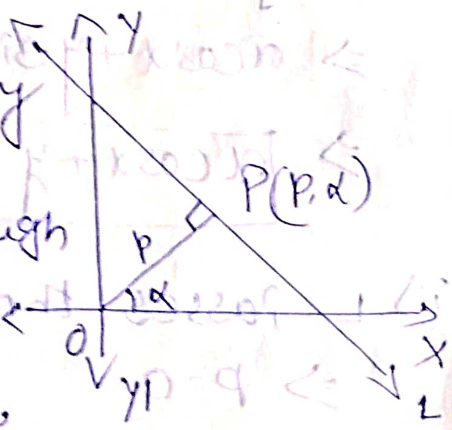
$$\Rightarrow y = \frac{b}{-a}(x-a)$$

$$\Rightarrow \frac{y}{b} = \frac{-x}{a} + 1$$

$$\Rightarrow \boxed{\frac{x}{a} + \frac{y}{b} = 1}$$

⑤ **Equation of a line in normal form:-**

Let a line L be at a distance p from the origin, and let the line through origin, $\perp^r$ to L, meet it at P. If P is the point (p,α) in polar co-ordinates, then



i) The equation of L is given by
$x\cos\alpha + y\sin\alpha = p$,
Provided L does not pass through origin.

ii) If L passes through origin,
$x\cos\alpha + y\sin\alpha = p$ is the equation of L
Provided $\alpha = \frac{\pi}{2} + \theta$, where θ is the inclination of L.

Proof:- i) L does not pass through the origin.
$\overleftrightarrow{OP}$ is the normal of the line.

Since P has polar co-ordinates $P(p, \alpha)$, it has Cartesian co-ordinates $(p\cos\alpha, p\sin\alpha)$.

Also origin $(0,0)$ is on $\overleftrightarrow{OP}$.

$\therefore$ slope of $\overleftrightarrow{OP} = \dfrac{p\sin\alpha - 0}{p\cos\alpha - 0} = \dfrac{p\sin\alpha}{p\cos\alpha} = \tan\alpha$

$\Rightarrow$ slope of $\overleftrightarrow{OP} = \tan\alpha$.

As $\overleftrightarrow{OP} \perp L \Rightarrow$ slope of $\overleftrightarrow{OP}$. slope of $L = -1$

$\Rightarrow \tan\alpha$. slope of $L = -1$

$\Rightarrow$ slope of $L = \dfrac{-1}{\tan\alpha} = -\cot\alpha$

Let $Q(x,y)$ be any point on L, then

slope of $L = \dfrac{y - p\sin\alpha}{x - p\cos\alpha}$

$\Rightarrow -\cot\alpha = \dfrac{y - p\sin\alpha}{x - p\cos\alpha}$

$\Rightarrow \dfrac{y - p\sin\alpha}{x - p\cos\alpha} = \dfrac{-\cos\alpha}{\sin\alpha}$

$\Rightarrow y\sin\alpha - p\sin^2\alpha = -x\cos\alpha + p\cos^2\alpha$

$\Rightarrow x\cos\alpha + y\sin\alpha = p(\sin^2\alpha + \cos^2\alpha)$

$\Rightarrow \boxed{x\cos\alpha + y\sin\alpha = p}$

ii) L passes through origin.

$\Rightarrow p = 0$

So polar co-ordinate of P be $(0, \alpha)$, where α is arbitrary and consequently

$$x\cos\alpha + y\sin\alpha = p = 0$$

which represents the family of lines passing through origin.

⑥ General Form :-

Let us consider the general equation of 1st degree in x and y given by -

$$Ax + By + C = 0 \quad \text{———} *$$

which is known as general eqn of Line.

If $A = 0 = B \Rightarrow 0.x + 0.y + C = 0 \Rightarrow C = 0.$

$\Rightarrow$ No equation is obtained.

If $A \neq 0$ and $B = 0 \Rightarrow Ax + C = 0 \Rightarrow \boxed{x = -C/A}$

which is the eqn to a line Parallel to Y-axis.

If $A = 0$ and $B \neq 0 \Rightarrow By + C = 0 \Rightarrow \boxed{y = -C/B}$

which is the eqn to a line Parallel to X-axis.

If $A \neq 0$ and $B \neq 0 \Rightarrow Ax + By = -C$

$$\Rightarrow By = -C - Ax$$

$$\Rightarrow \boxed{y = \left(-\frac{A}{B}\right)x + \left(-\frac{C}{B}\right)}$$

which is in slope - Intercept form.

Thus, slope $(m) = -\frac{A}{B}$ and y-intercept $= -\frac{C}{B}$

Again, $Ax + By + C = 0$

$$\Rightarrow Ax + By = -C$$

$$\Rightarrow \left(-\frac{A}{C}\right)x + \left(-\frac{B}{C}\right)y = 1$$

$$\Rightarrow \boxed{\frac{x}{(-C/A)} + \frac{y}{(-C/B)} = 1}$$

which is in Intercept form.

Thus, x-intercept $= -C/A$

Note :- If $Ax + By + C = 0$ be the eqn of line, then
* slope $(m) = -A/B$
* x-intercept $(a) = \frac{-C}{A}$
* y-intercept $(b) = \frac{-C}{B}$

Lines continued :-

Consider the eqn's of lines L_1 and L_2 :-

$L_1 : a_1x + b_1y + c_1 = 0$
$L_2 : a_2x + b_2y + c_2 = 0$

$m_1 =$ slope of $L_1 \Rightarrow m_1 = -\frac{a_1}{b_1}$

$m_2 =$ slope of $L_2 \Rightarrow m_2 = -\frac{a_2}{b_2}$

suppose L_1 and L_2 neither be vertical nor horizontal.

i.e $a_1 \neq 0$, $a_2 \neq 0$, $b_1 \neq 0$ and $b_2 \neq 0$.

Parallel lines :-

The lines L_1 and L_2 are parallel if and only if - $\boxed{m_1 = m_2}$

$$\Rightarrow -\frac{a_1}{b_1} = -\frac{a_2}{b_2} \Rightarrow \frac{a_1}{b_1} = \frac{a_2}{b_2} \Rightarrow \boxed{\frac{a_1}{a_2} = \frac{b_1}{b_2}}$$

OR, $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{1}{\lambda}$ (Let)

$\Rightarrow a_2 = \lambda a_1$ and $b_2 = \lambda b_1$

$L_2 : a_2x + b_2y + c_2 = 0 \Rightarrow \lambda a_1x + \lambda b_1y + c_2 = 0$

$\Rightarrow \lambda(a_1x + b_1y + c_2/\lambda) = 0$

$\Rightarrow a_1x + b_1y + c_3 = 0$, $c_3 = c_2/\lambda$

If a line L is represented by $ax + by + c = 0$ then the equation of a line L', Parallel to L is given by – $ax + by + d = 0$.

Perpendicular lines:-

The lines L_1 and L_2 are perpendicular iff

$$\boxed{m_1 \cdot m_2 = -1}$$

$$\Rightarrow -\frac{a_1}{b_1} \cdot -\frac{a_2}{b_2} = -1$$

$$\Rightarrow a_1 a_2 = -b_1 b_2$$

$$\Rightarrow \boxed{\frac{a_1}{b_1} = -\frac{b_2}{a_2}}$$

If a line L is represented by $ax + by + c = 0$ then the eq^n of line L', perpendicular to L is given by-

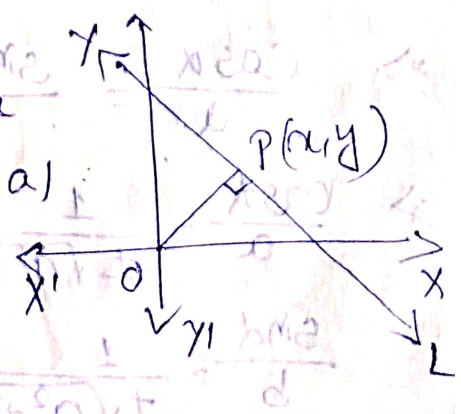
$$bx - ay + d = 0$$

~~Con~~ Coincident lines:-

The lines L_1 and L_2 are coincidence iff

$$\boxed{\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}}$$

Length of Perpendicular from origin on a line :-



- For vertical line $x = a$ i.e. a ~~to~~ ⊥r distance from origin $= |a|$
- For horizontal line $y = b$ ⊥r distance from origin $= |b|$

Let L be the oblique line.

$L: ax+by+c=0$. with $a, b \neq 0$.

Let $c=0 \Rightarrow ax+by=0$

It is obvious that origin is a point on the line.

Thus the perpendicular distance from origin to the line be zero.

Let $c \neq 0$ and p denotes the $\perp$ distance from origin -

The eqn of line in normal form -

$$x\cos\alpha + y\sin\alpha = p$$

$$\Rightarrow x\cos\alpha + y\sin\alpha - p = 0$$

$L: ax+by+c=0$.

$L: x\cos\alpha + y\sin\alpha - p = 0$.

Since the line is same, the condition of coincident works -

$$\frac{\cos\alpha}{a} = \frac{\sin\alpha}{b} = \frac{-p}{c}$$

Now, $\dfrac{\sqrt{\cos^2\alpha + \sin^2\alpha}}{\pm\sqrt{a^2+b^2}} = \dfrac{1}{\pm\sqrt{a^2+b^2}}$

$$\Rightarrow \frac{\cos\alpha}{a} = \frac{\sin\alpha}{b} = -\frac{p}{c} = \frac{1}{\pm\sqrt{a^2+b^2}}$$

$$\Rightarrow \frac{\cos\alpha}{a} = \frac{1}{\pm\sqrt{a^2+b^2}} \Rightarrow \cos\alpha = \frac{a}{\pm\sqrt{a^2+b^2}}$$

$$\frac{\sin\alpha}{b} = \frac{1}{\pm\sqrt{a^2+b^2}} \Rightarrow \sin\alpha = \frac{b}{\pm\sqrt{a^2+b^2}}$$

$$\frac{-p}{c} = \frac{1}{\pm\sqrt{a^2+b^2}} \Rightarrow p = \frac{-c}{\pm\sqrt{a^2+b^2}}$$

Thus the perpendicular distance from the origin to the line be -

$$\boxed{p = \frac{-c}{\pm\sqrt{a^2+b^2}}}$$

Since $p>0$, the sign of the radical in the denominator has to be $+$ or $-$ accordingly as $c<0$ or $c>0$.

Point of Intersection :-

If two distinct lines L_1 and L_2 represented by -

$L_1 : a_1x + b_1y + c_1 = 0$

$L_2 : a_2x + b_2y + c_2 = 0.$

intersects at $P(h,k)$, then

$$\boxed{h = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1} \quad \text{and} \quad k = \frac{c_1a_2 - a_1c_2}{a_1b_2 - a_2b_1}}$$

Concurrent lines :

If three lines L_1, L_2 and L_3 be represented by -

$L_1 : a_1x + b_1y + c_1 = 0$

$L_2 : a_2x + b_2y + c_2 = 0$

$L_3 : a_3x + b_3y + c_3 = 0$, then

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

$\Leftrightarrow$ L_1, L_2 and L_3 are all Parallel or concurrent.

Family of Lines through the Point of intersection of two Lines :-

Let $L_a : a_1x + b_1y + c_1 = 0$

$L_b : a_2x + b_2y + c_2 = 0$

Now consider the eqn -

$\boxed{L_a + \lambda L_b = 0}$, where λ is a constant.

$\Rightarrow (a_1x + b_1y + c_1) + \lambda(a_2x + b_2y + c_2) = 0$

$\Rightarrow (a_1 + \lambda a_2)x + (b_1 + \lambda b_2)y + (c_1 + \lambda c_2) = 0$

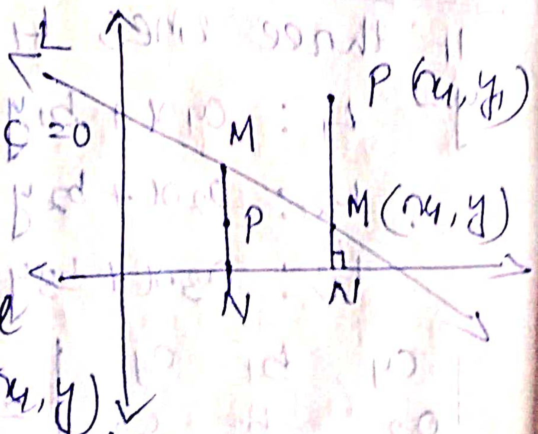
This is called the family of lines, where $\lambda \in R$.

Theorem:-

i) $(L_a + \lambda L_b)$ represents the family of lines through the point of intersection of L_a and L_b if they intersects.

ii) $(L_a + \lambda L_b)$ represents the family of lines Parallel to L_a and L_b if they are Parallel.

Distance of a Point from a Line :-



Consider a non-vertical line L given by $L : ax + by + c = 0$ and a point $P(x_1, y_1)$.

Let a line through P parallel to Y-axis, intersects at $M(x_1, y)$.

i) $P(x_1, y_1)$ is above $L \iff y_1 > y \Rightarrow y_1 - y > 0$.

ii) $P(x_1, y_1)$ is below $L \iff y_1 < y \Rightarrow y_1 - y < 0$.

$y_1 - y$ = Distance between the point and line.

$M(x_1, y)$ is on the line L.

$$a x_1 + b y + c = 0.$$

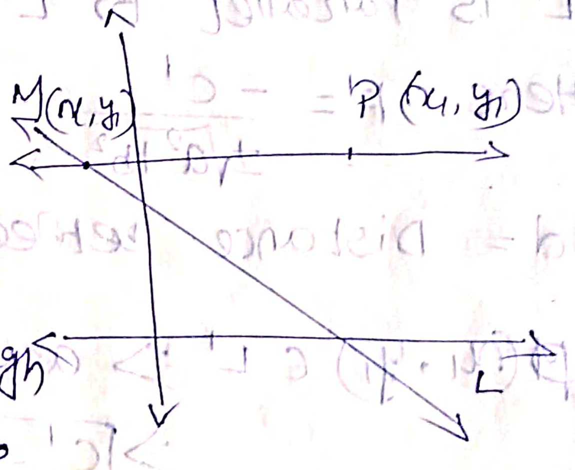
$$\Rightarrow b y = -(c + a x_1)$$

$$\Rightarrow y = -\left(\frac{a x_1 + c}{b}\right)$$

$$\therefore y_1 - y = y_1 + \left(\frac{a x_1 + c}{b}\right) = \frac{a x_1 + b y_1 + c}{b}$$

Hence $P(x_1, y_1)$ will lie above or below the line $L: ax + by + c = 0$ if $y_1 - y = \frac{a x_1 + b y_1 + c}{b}$ is positive or negative.

Consider a non-horizontal line $L: ax + by + c = 0$. and $P(x_1, y_1)$.



Let a line passing through P, parallel to x-axis, intersects L at the point (x_1, y_1)

i) $P(x_1, y_1)$ is to the right of L

$\iff x_1 > x$ i.e $x_1 - x > 0$

ii) $P(x_1, y_1)$ is to the left of L

$\iff x_1 < x$ i.e $x_1 - x < 0$

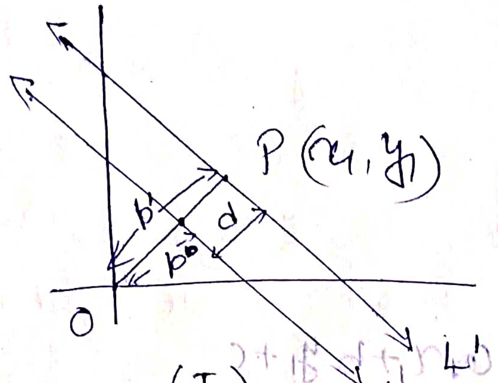
$x_1 - x$ = Distance between the point and L.

$M(x, y_1)$ is on $L \Rightarrow ax + by_1 + c = 0$.

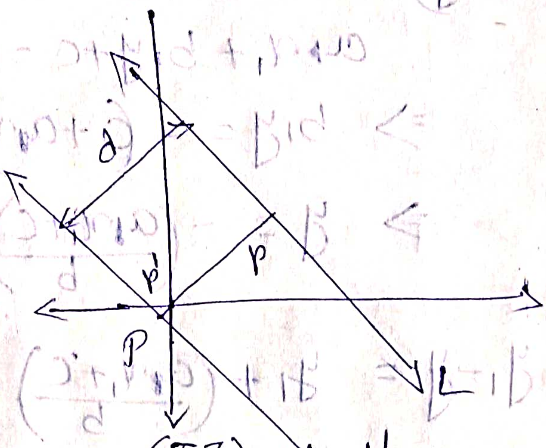
$\therefore x = -\left(\frac{by_1+c}{a}\right)$

$\Rightarrow x_1 - x = x_1 + \left(\frac{by_1+c}{a}\right) = \frac{ax_1+by_1+c}{a}$

Normal form:-



(I)



(II)

L : $ax+by+c=0$, then the distance i.e normal $p = \frac{-c}{\pm\sqrt{a^2+b^2}}$

L' is parallel to L $\Rightarrow$ L': $ax+by+c'=0$.

Here $p' = \frac{-c'}{\pm\sqrt{a^2+b^2}}$

d = Distance between point and line.

$P(x_1, y_1) \in L' \Rightarrow ax_1+by_1+c'=0$

$\Rightarrow \boxed{c' = -(ax_1+by_1)}$

In case (I)

$d = |p - p'| = \left| \frac{-c}{\pm\sqrt{a^2+b^2}} - \frac{-c'}{\pm\sqrt{a^2+b^2}} \right|$

$= \left| \frac{-c+c'}{\pm\sqrt{a^2+b^2}} \right|$

$= \left| \frac{-c+(-ax_1-by_1)}{\pm\sqrt{a^2+b^2}} \right| = \left| \frac{-(ax_1+by_1+c)}{\pm\sqrt{a^2+b^2}} \right|$

$$\Rightarrow d = \frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}$$

In Case-II :-

$$d = |P+P'| = \left|\frac{-c}{\pm\sqrt{a^2+b^2}} + \frac{-c'}{\mp\sqrt{a^2+b^2}}\right| = \left|\frac{-c}{\pm\sqrt{a^2+b^2}} + \frac{c'}{\pm\sqrt{a^2+b^2}}\right|$$

$$= \left|\frac{c'-c}{\pm\sqrt{a^2+b^2}}\right|$$

$$= \left|\frac{ax_1-by_1-c}{\pm\sqrt{a^2+b^2}}\right| = \left|\frac{ax_1+by_1+c}{\pm\sqrt{a^2+b^2}}\right|$$

$$\Rightarrow d = \frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}$$

Thus, the distance of a Point $P(x_1, y_1)$ from the line $L: ax+by+c=0$ is given by -

$$\boxed{d = \frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}}$$

Equation of Angle bisector :-

$L_1: a_1x+b_1y+c_1=0$

$L_2: a_2x+b_2y+c_2=0$

B_1 be the angle bisector between L_1 and L_2.

Let $P(x, y)$ on the B_1.

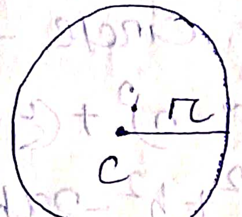
Then eqn of two bisectors will be -

$$\boxed{\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}} = \pm\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}}$$

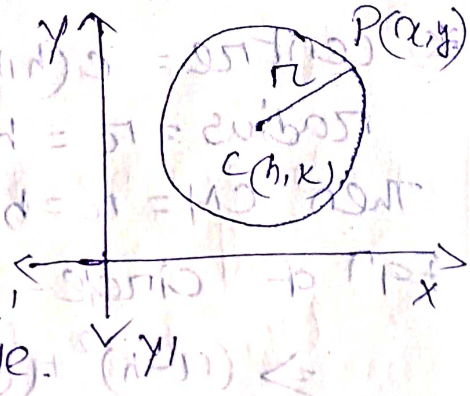
CIRCLES

Circle :- A circle is the locus of all points in a plane which are equidistance from a given point in that ~~Point~~ plane.

The given point is called the centre of the circle.



A line segment joining the centre to a point on the circle as well as its length is known as the radius of the circle.



Equation of circle with given centre and radius :-

Let C(h,k) be the centre and r be the radius of the circle.

Let P(x,y) be any point on the circle.

$r = |CP| = \sqrt{(x-h)^2 + (y-k)^2}$

$\Rightarrow r^2 = CP^2 = (x-h)^2 + (y-k)^2$

$\therefore \boxed{(x-h)^2 + (y-k)^2 = r^2}$

which is the required eqn of the circle.

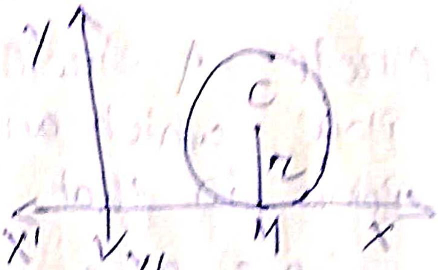
If the centre is at origin O (0,0), then the eqn of circle be

$(x-0)^2 + (y-0)^2 = r^2$

$\Rightarrow \boxed{x^2 + y^2 = r^2}$

Notes :-

* centre = C(h,k)
  Radius = r = k
  CM = r ⇒ Let CM = k



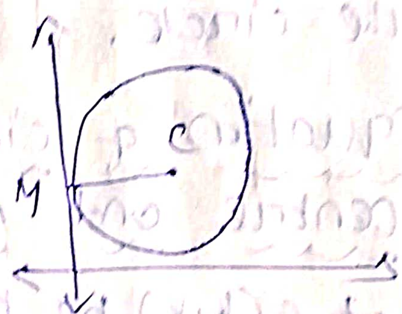
Eqn of circle - $(x-h)^2+(y-k)^2=r^2$

$\Rightarrow (x-h)^2+(y-k)^2=k^2$

$\Rightarrow x^2+h^2-2xh+y^2+k^2-2ky=k^2$

$\Rightarrow \boxed{x^2+y^2-2xh-2yk+h^2=0}$

* centre = C(h,k)
  radius = r = h
  Then CM = r = h

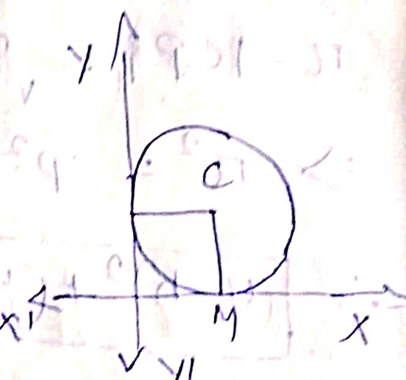


Eqn of circle - $(x-h)^2+(y-k)^2=r^2$

$\Rightarrow (x-h)^2+(y-k)^2=h^2$

$\Rightarrow \boxed{x^2+y^2-2xh-2yk+k^2=0}$

* Radius = r = h = k = CM

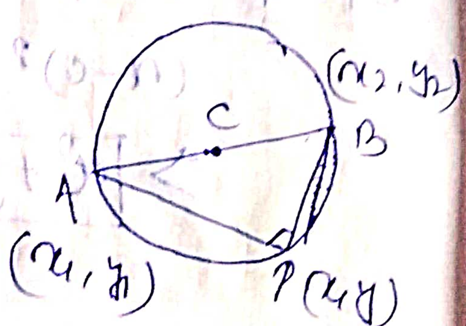


$(x-h)^2+(y-k)^2=h^2=k^2$

$\Rightarrow \boxed{\begin{array}{l} x^2+y^2-2xh-2yh+h^2=0 \text{ or} \\ x^2+y^2-2xk-2yk+k^2=0 \end{array}}$

Circle on a given Diameter :-

Let A(x_1,y_1), B(x_2,y_2) be the end points of the diameter.



Let P(x,y) be another point on the circle.

Then $\angle APB = \pi/2$ ⇒ Δ APB be a right angled triangle.

$\Rightarrow AP \perp BP$

$\Rightarrow$ (slope of $\overleftrightarrow{AP}$) (slope of $\overleftrightarrow{BP}$) $= -1$

$$\Rightarrow \left(\frac{y-y_1}{x-x_1}\right)\left(\frac{y-y_2}{x-x_2}\right) = -1$$

$$\Rightarrow (y-y_1)(y-y_2) = -(x-x_1)(x-x_2)$$

$$\Rightarrow \boxed{(x-x_1)(x-x_2) + (y-y_1)(y-y_2) = 0}$$

which is the equation of the circle.

General Equation of a circle :-

Let centre be $C(h,k)$ and radius be r, then eqn of circle- $(x-h)^2 + (y-k)^2 = r^2$

$$\Rightarrow x^2 + h^2 - 2xh + y^2 - 2ky + k^2 = r^2$$

$$\Rightarrow x^2 + y^2 - 2hx - 2ky + (h^2 + k^2 - r^2) = 0 \quad \text{---} (1)$$

which is of the form-

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{---} (2)$$

Then from eqn (1) and (2), we get-

$g = -h \Rightarrow h = -g$ and $f = -k \Rightarrow k = -f$

$c = h^2 + k^2 - r^2 = (-g)^2 + (-f)^2 - r^2 = g^2 + f^2 - r^2$

$\Rightarrow r^2 = -c + g^2 + f^2 \cancel{\Rightarrow r = \pm c - g^2 + f}$

$\Rightarrow r = \sqrt{g^2 + f^2 - c}$

$$\therefore \boxed{\begin{array}{l} \text{centre, } C(h,k) = C(-g, -f) \\ \text{radius, } r = \sqrt{g^2 + f^2 - c} \end{array}}$$