LEARNING MATERIAL

ON

CO-ORDINATE GEOMETRY IN 2D

SEMESTER : I
DEPARTMENT: MATHEMATICS AND SCIENCE
SUBJECT NAME: ENGINEERING MATHEMATICS-I
SUBJECT CODE: TH.3

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STRAIGHT LINES

Length of perpendicular distance from a point to the line and perpendicular distance between parallel lines. Simple problems.

Angle between two straight lines and condition for parallel and perpendicular lines. Simple problems

Pair of straight lines Through origin

Pair of lines passing through the origin $ax^2+2hxy+by^2=0$ expressed in the form $(y-m_1x)(y-m_2x)=0$. Derivation of

$\tan\theta = +\dfrac{2\sqrt{h^2-ab}}{a+b}$ condition for parallel and perpendicular lines. Simple problems.

Pair of straight lines not through origin

Condition for general equation of the second degree $ax^2+2hxy+by^2+2gx+2fy+c=0$ to represent pair of lines.

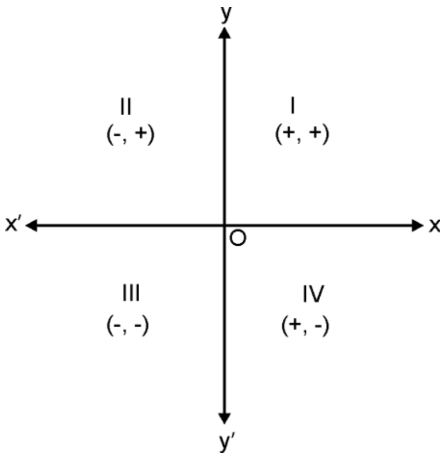
$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0 \quad \text{(Statement only)}$$

Angle between them, condition for parallel and perpendicular lines simple problems.

STRAIGHT LINES

Introduction

Analytical Geometry is a branch of Mathematics which deals with solutions of geometrical problems by Algebraic methods. It was developed by the famous French mathematician called Rune Descartes.

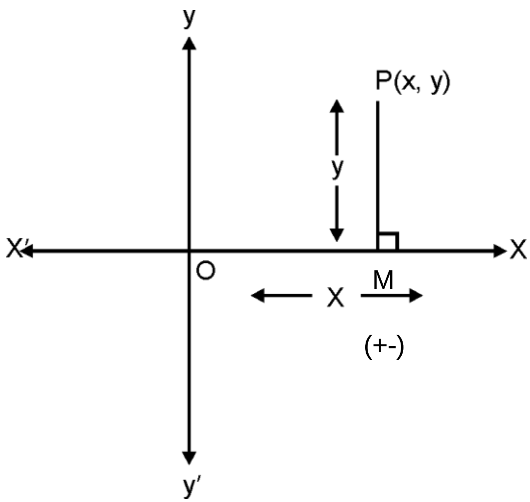


Axes of co-ordinates:

Take two straight lines XOX' and YOY' at right angles to each other. The horizontal line XOX' is called the X – axis and the vertical line YOY' is called the Y-axis. These two axes intersect at O, called the origin.

Cartesian – Rectangular Co-ordinates:

Diagram



Let XOX’ and YOY’ be the axes of co-ordinates. Let P be any point in the plane. Draw PM perpendicular to OX. Then the position of P is uniquely determined by the distances OM and MP. These distances OM and MP are called the Cartesian rectangular co-ordinates of the point P with respect to X-axis and Y–axis respectively.

It is to be noted that the ‘X’ co-ordinate must be in first place and the ‘Y’ co-ordinate must be in the second place. This order must be strictly followed.

Straight Line:

When a variable point moves in accordance with a geometrical law, the point will trace some curve. This curve is known as the locus of the variable point.

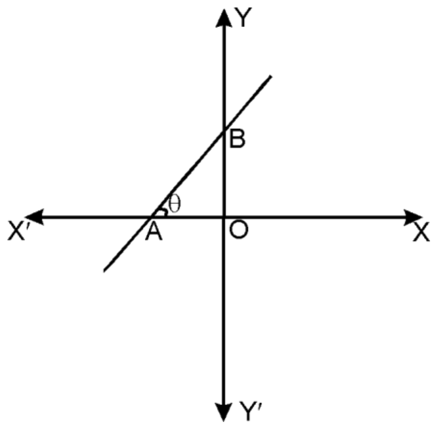
If a relation in x and y represent a curve then

(i) The co-ordinates of every point on the curve will satisfy the relation.

(ii) Any point whose co-ordinates satisfy the relation will lie on the curve.

Straight line is a locus of a point.

Diagram



Let the line AB cut the X-axis at A and y-axis at B. The angle made by the line AB with the positive direction of the x-axis is called the angle of inclination of the line AB with the x-axis and it is denoted by θ . Hence $\angle XAB = \theta$ .The angle can take any values from 0° to 180°.

Slope or gradient of a straight line:

The tangent of the angle of inclination of the straight line is called slope or gradient of the line. If θ is the angle of inclination then

slope = $\tan\theta$ and is denoted by m.

Example:

If a line makes an angle of 45° with the X-axis in the positive direction then the slope of the line is $\tan 45^\circ$.

i.e $m = \tan 45^\circ = 1$

In school studies students have learnt, the distance between two points section formula, mid point of the line joining two points, various form of equation of the straight line, point of intersection of two lines, etc., in analytical Geometry.

Standard forms of the equation of a straight line.

(i) Slope – intercept form:

When 'c' is the y intercept and slope is 'm', the equation of the straight line is $y=mx+c$

(ii) Slope – point form:

When 'm' is the slope of the straight line and (x_1,y_1) is a point on the straight line its equation is $y-y_1 = m(x-x_1)$

(iii) Two – point form:

Equation of the line joining the two points (x_1,y_1) and (x_2,y_2) is

$$\frac{y-y_1}{y_1-y_2} = \frac{x-x_1}{x_1-x_2}$$

(iv) Intercept form:

When the x and y intercepts of a straight line are given as ‘a’ and ‘b’ respectively, the equation of the straight line is

ie., $\frac{x}{a} + \frac{y}{b} = 1$

(v). **General form:**

The general form of the equation of a straight line is ax+by+c = 0. If ax+by+c=0 is the equation of a straight line then

$$\text{Slope } m = -\frac{\text{coefficient of } x}{\text{coefficient of } y} = -\frac{a}{b}$$

$$x\text{-intercept} = -\frac{\text{constant term}}{\text{coefficient of } x} = -\frac{c}{a}$$

$$y\text{-intercept} = -\frac{\text{constant term}}{\text{coefficient of } y} = -\frac{c}{b}$$

Some Important Formulae:

(i) The length of the perpendicular from (x_1,y_1) to the line

ax + by + c = 0 is

$$\pm\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}$$

(ii) The length of the perpendicular from origin to the line

ax + by + c = 0 is

$$\pm\frac{c}{\sqrt{a^2 + b^2}}$$

(iii) The distance between the parallel lines $ax + by + c_1 = 0$ and

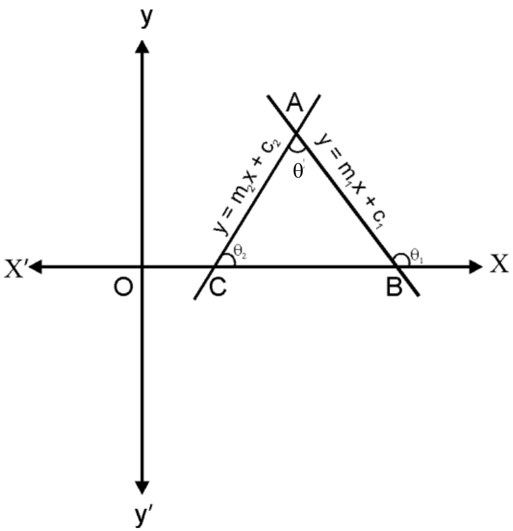
$ax + by + c_2 = 0$ is

$$\pm\frac{c_1 - c_2}{\sqrt{a^2 + b^2}}$$

ANGLE BETWEEN TWO STRAIGHT LINES

Book Work:

Find the angle between the lines $y = m_1x+c_1$ and $y = m_2x+c_2$. Deduce the conditions for the lines to be (i) parallel (ii) perpendicular



Proof:

Let θ_1 be the angle of inclination of the line $y = m_1x + c_1$. Slope of this line is $m_1 = \tan\theta_1$. Let 'θ_2' be the angle of inclination of the line $y = m_2x + c_2$. Slope of this line is $m_2 = \tan\theta_2$.

Let 'θ' be the angle between the two lines, then $\theta_1 = \theta_2 + \theta \rightarrow \theta=\theta_1-\theta_2$

$$\therefore \tan\theta = \tan(\theta_1-\theta_2)$$

$$= \frac{\tan\theta_1 - \tan\theta_2}{1+\tan\theta_1\tan\theta_2}$$

$$\tan\theta = \frac{m_1 - m_2}{1+m_1m_2}$$

$$\rightarrow \theta = \tan^{-1}\left(\frac{m_1 - m_2}{1+m_1m_2}\right)$$

(i) Condition for two lines to be parallel:

If the two lines are parallel then the angle between the two lines is zero

$$\therefore \tan\theta = \tan 0 = 0$$

(i.e) $$\frac{m_1 - m_2}{1 + m_1 m_2} = 0$$

$$m_1 - m_2 = 0$$

$$\therefore m_1 = m_2$$

$\therefore$ For parallel lines, slopes are equal.

(ii) Condition for two lines to be perpendicular:

If the two lines are perpendicular then the angle between them $\theta = 90°$

$$\therefore \tan\theta = \tan 90° = \infty = \frac{1}{0}$$

$$\therefore \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{1}{0}$$

$$1 + m_1 m_2 = 0$$

$$m_1 m_2 = -1$$

$\therefore$ For perpendicular lines, product of the slopes will be -1

Note :

1) The acute angle between the lines

$Y = m_1 x + c_1$ and $y = m_2 x + c_2$ is $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

2) If the slope of a line is 'm' then the slope of parallel line is also m.

3) If the slope of a line is m then the slope of any line perpendicular to the line is $-\frac{1}{m}$

4) Any line parallel to the line ax+by+c = 0 will be of the form ax+by+d = 0 (differ only constant term)

5) Any line perpendicular the line ax+by+c =0 will be of the form bx-ay+d=0

WORKED EXAMPLESPART – A

1) Find the perpendicular distance from the point (2,3) to the straight line 2x+y+3=0

Solution:

The length of the perpendicular from the point (x_1,y_1) to the line ax+by+c = is

$$\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}$$

Give straight line is 2x+y+3 =0

Given point (x_1,y_1) = (2,3)

(i.e) $\frac{2(2)+(3)+3}{\sqrt{(2)^2+(1)^2}} = \frac{10}{\sqrt{5}}$

2) Find the length of the perpendicular to the line 4x+6y+7 = 0 from the origin

Solution:

The length of the perpendicular is $\frac{c}{\sqrt{a^2+b^2}}$

Here a = 4, b=6, c =7

(i.e.) $\frac{7}{\sqrt{(4)^2+(6)^2}} = \frac{7}{\sqrt{16+36}} = \frac{7}{\sqrt{52}}$

3) Find the distance between the line 2x+3y+4 = 0 and 2x+3y -1 = 0

Solution:

The distance between the parallel lines is $\frac{|c_1 - c_2|}{\sqrt{a^2 + b^2}}$

Here c_1 = 4 and c_2 = -1

Now $\frac{|4+1|}{\sqrt{(2)^2 + (-3)^2}}$

$= \frac{5}{\sqrt{13}}$

4) Find the angle between the lines $y = \sqrt{3}x$ and x-y = 0

Solution:

$y = \sqrt{3}x$

(i.e.) $\sqrt{3}x - y = 0$ (1) and x-y =0 (2)

Slope of (1) = $-\frac{\text{coefficient of x}}{\text{coefficient of y}}$

$= \frac{-\sqrt{3}}{-1} = \sqrt{3}$

$\tan \theta_1 = \sqrt{3} \rightarrow \theta_1 = 60°$

slope of (2) = $-\frac{\text{coefficient of x}}{\text{coefficient of y}}$

$\tan \theta_2 = -\frac{1}{-1} = 1$

$\theta_2 = 45°$

Let θ be the angle between (1) and (2)

$\therefore \theta = \theta_1 - \theta_2$

$\theta = 60° - 45° = 15°$

5) Show that the lines $6x+y-11=0$ and $12x+2y+14=0$ are parallel

Solution:

$$6x+y-11 = 0 \qquad (1)$$

$$12x+2y+14 = 0 \qquad (2)$$

Slope of the line (1) = $m_1 = -\frac{a}{b} = -\frac{6}{1} = -6$

Slope of the line (2) = $m_2 = -\frac{a}{b} = -\frac{12}{2} = -6$

$m_1 = m_2$

$\therefore$ The lines are parallel.

6) Find 'p' such that the lines $7x-4y+13 = 0$ and $px = 4y+6$ are parallel.

Solution:

$$7x-4y+13=0 \qquad (1)$$

$$px-4y-6 = 0 \qquad (2)$$

Slope of the line (1) $m_1 = \frac{-a}{b} = \frac{-7}{-4} = \frac{7}{4}$

Slope of the line (2) $m_2 = \frac{-a}{b} = \frac{-p}{-4} = \frac{p}{4}$

Since (1) and (2) are parallel lines

$m_1 = m_2$

$$\frac{7}{4} = \frac{p}{4}$$

$4p = 28$

$$p = \frac{28}{4} = 7$$

$\therefore$ $p = 7$

7) Show that the lines 2x+3y-7 =0 and 3x-2y+4=0 are perpendicular.

Solution:

$2x+3y-7 = 0$ (1)

$3x-2y+4 = 0$ (2)

Slope of the line (1) $= m_1 = \frac{-2}{3}$

Slope of the line (2) $= m_2 = \frac{-3}{-2} = \frac{3}{2}$

Now $m_1 m_2 = \left(\frac{-2}{3}\right)\left(\frac{3}{2}\right) = -1$

$\therefore m_1 m_2 = -1$

$\therefore$ The lines (1) and (2) are perpendicular

8) Find the value of m if the lines 2x+my=4 and x+5y-6=0 are perpendicular

Solution:

$2x+my-4=0$ (1)

$x+5y-6 = 0$ (2)

Slope of the line (1) $= m_1 = -\frac{2}{m}$

Slope of the line (2) $= m_2 = -\frac{1}{5}$

Since the liner are perpendicular

$m_1 m_2 = -1$

$\left(-\frac{2}{m}\right)\left(-\frac{1}{5}\right) = -1$

$\frac{2}{5m} = -1$

$-5m = 2$

$m = -\frac{2}{5}$

PART – B

1) Find the angle between the lines 3x+6y=8 and 2x = -y+5

Solution:

$3x+6y-8 = 0$ (1)

Slope of the line (1) $= m_1 = \frac{-a}{b} = \frac{-3}{6} = \frac{-1}{2}$

$2x+y-5 = 0$ (2)

Slope of the line (2) $= m_2 = \frac{-a}{b} = \frac{-2}{1} = -2$

Let 'θ' be the angle between two lines

$$\tan\theta = \left|\frac{m_1 - m_2}{1+m_1m_2}\right|$$

$$= \left|\frac{-\frac{1}{2}+2}{1+\left(\frac{-1}{2}\right)(-2)}\right| = \left|\frac{\frac{3}{2}}{2}\right| = \left|\frac{3}{4}\right| = 0.75$$

$\therefore \theta = \tan^{-1}(0.75)$

$\rightarrow \theta = 36^\circ\, 52'$

2) Find the equation of the straight line passing through (-1, 4) and parallel to x+2y =3.

Solution:

Let the equation of line parallel to $x+2y-3 = 0$ (1)

is $x+2y+k = 0$ (2)

Equation (2) passes through (-1,4)

put x=-1, y =4 in equation (2)

$(-1) + 2(4) + k = 0$

$-1 + 8 + k = 0$

$k = -7$

$\therefore$ Required line is $x + 2y - 7 = 0$

3) Find the equation to the line through the point (3,-3) and perpendicular to $4x-3y-10=0$

Solution:

Required straight line is perpendicular to $4x-3y-10=0$ (1)

and passing through (3,-3).

$\therefore$ Required equation of the straight line is

$$-3x - 4y + k = 0 \qquad (2)$$

Required line passes through (3,-3)

Put $x = 3$, $y = -3$ in equation (2)

$$-3(3) - 4(-3) + k = 0$$

$$-9+12+k=0$$

$$3+k=0$$

$$k=-3$$

Sub in equation (2)

$$-3x - 4y - 3 = 0$$

$\therefore$ Required equation of straight line is

$$3x + 4y + 3 = 0$$

PAIR OF STRAIGHT LINES THROUGH ORIGIN

Any line passing through the origin is of the form $ax+by=0$

Let $a_1x + b_1y = 0$ (1)

and $a_2x + b_2y = 0$ (2)

be the two lines passing through the origin.

The combined equation of (1) and (2) is

$$(a_1x+b_1y)(a_2x+b_2y) = 0$$

$$a_1a_2x^2 + (a_1b_2+a_2b_1)xy + b_1b_2y^2 = 0 \qquad (3)$$

Taking $a_1a_2 = a$, $a_1b_2+a_2b_1 = 2h$, and $b_1b_2 = b$

We get

$$ax^2 + 2hxy + by^2 = 0 \qquad (4)$$

which is a homogenous equation of second degree in x and y. It represents a pair of straight lines passing through the origin.

Let m_1 and m_2 are the slopes of the lines given by (4). Then the separate equations are

$y = m_1 x$ and $y = m_2 x$

(i.e.) $y - m_1 x = 0$ (5)

$y - m_2 x = 0$ (6)

$(y - m_1 x)(y - m_2 x) = 0$

$y^2 - (m_1+m_2)\,xy + m_1 m_2 x^2 = 0$

(i.e.) $m_1 m_2 x^2 - (m_1+m_2)xy + y^2 = 0$ (7)

Equation (4) and (7) represent the same part of straight lines. Hence the ratios of the corresponding co-efficient of like terms are proportional.

$$\frac{m_1 m_2}{a} = \frac{-(m_1+m_2)}{2h} = \frac{1}{b} \qquad (8)$$

The relation (8) gives

$$m_1+m_2 = \frac{-2h}{b} \qquad (9)$$

i.e., Sum of the slopes = $\frac{-2h}{b}$

and $m_1 m_2 = \frac{a}{b}$

i.e., product of the slopes = $\frac{a}{b}$

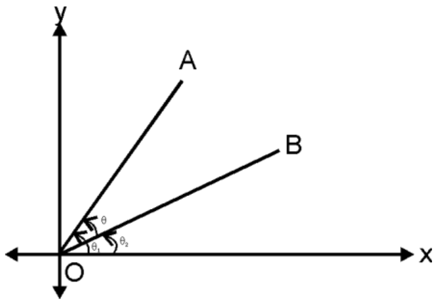
BOOK WORK :

Find the angle between the pair of straight lines $ax^2 + 2hxy + by^2 = 0$ passing through origin. Also derive the conditions for the two separate lines to be (i) perpendicular (ii) coincident (or parallel).

Proof:

We know angle between two straight lines is given by

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$



$$\tan\theta = \pm\frac{\sqrt{(m_1 + m_2)^2 - 4m_1 m_2}}{1 + m_1 m_2}$$

$$= \pm\frac{\sqrt{\left[\frac{-2h}{b}\right]^2 - 4\left[\frac{a}{b}\right]}}{1 + \frac{a}{b}}$$

$$= \pm\frac{\sqrt{\frac{4h^2}{b^2} - 4\left[\frac{a}{b}\right]}}{\frac{a+b}{b}}$$

$$= \pm\frac{\sqrt{\frac{4h^2 - 4ab}{b^2}}}{\frac{a+b}{b}}$$

$$= \pm\frac{\sqrt{4(h^2 - ab)}}{b} \text{ x } \frac{b}{a+b}$$

$\tan\theta = \pm 2\frac{\sqrt{h^2-ab}}{a+b}$

(i.e.) $\theta = \tan^{-1}\left[\pm 2\frac{\sqrt{h^2-ab}}{a+b}\right]$

is the angle between the pair of straight lines.

(iii) Condition for the two straight lines to be perpendicular

If the two lines are perpendicular, then $\theta = 90°$

$\therefore \tan\theta = \tan 90°$

$\pm 2\frac{\sqrt{h^2-ab}}{a+b} = \infty = \frac{1}{0}$

$\therefore a + b = 0$

(i.e.) coefficient of x^2 + coefficient of y^2 = 0

(iv) Condition for the two straight lines to be coincident

If the two straight lines are coincident

then $\theta = 0$

$\therefore \tan\theta = \tan 0$

$\pm 2\frac{\sqrt{h^2-ab}}{a+b} = 0$

(i.e.) $h^2 - ab = 0$

(i.e.) $h^2 = ab$

3.2 WORKED EXAMPLES

PART - A

1) Write down the combined equation of the pair of lines x-2y =0 and 3x+2y=0

Solution:

The combined equation is $(x-2y)(3x+2y) = 0$

(i.e.) $3x^2 + 2xy - 6xy - 4y^2 = 0$

(i.e.) $3x^2 - 4xy - 4y^2 = 0$

2) Write down the separate equations of the pair of lines $12x^2 + 7xy - 10y^2 = 0$

Solution:

$12x^2 + 7xy - 10y^2 = 0$

$12x^2 + 15xy - 8xy - 10y^2 = 0$

$12x^2 - 8xy + 15xy - 10y^2 = 0$

$4x(3x-2y) + 5y(3x-2y) = 0$

$(3x-2y)(4x+5y) = 0$

$\therefore$ The separate equations are $3x-2y = 0$ and $4x+5y = 0$

3) Show that the two lines represented by $4x^2 + 4xy + y^2 = 0$ are parallel to each other.

Solution:

$$4x^2 + 4xy + y^2 = 0 \qquad (1)$$

This is of the form $ax^2 + 2hxy + by^2 = 0$

Here a =4, 2h =4, h=2, b=1

If the lines are parallel then $h^2 - ab = 0$

$$\begin{aligned} h^2 - ab &= 2^2 - (4)(1) \\ &= 4 - 4 = 0 \end{aligned}$$

$\therefore$ pair of lines are parallel

4) Find the value of 'p' if the pair of lines $4x^2 + pxy + 9y^2 = 0$ are parallel to each other.

Solution:

$$4x^2 + pxy + 9y^2 = 0$$

This is of the form $ax^2+2hxy+by^2=0$

Here $a = 4$, $2h = p$, $h = \frac{p}{2}$, $b = 9$

If the lines are parallel

$$h^2 - ab = 0$$

$$\left(\frac{p}{2}\right)^2 - (4)(9) = 0$$

$$\frac{p^2}{4} - 36 = 0$$

$$p^2 = 144$$

$$\therefore\ p = \pm 12$$

5) Prove that the lines represented by $7x^2-48xy-7y^2 = 0$ are perpendicular to each other.

Solution:

$$7x^2 - 48xy - 7y^2 = 0$$

This is of the form $ax^2+2hxy+by^2=0$

Here $a = 7$, $2h = -48$, $h = -24$, $b = -7$

If the lines are perpendicular $a+b=0$

(i.e.) $7-7 = 0$

6) If the two straight lines represented by the equation $px^2-5xy+7y^2 = 0$ are perpendicular to each other, find the value of p.

$px^2+48xy+7y^2 = 0$

Solution:

$$px^2+48xy+7y^2 = 0$$

This is of the form $ax^2+2hxy+by^2=0$

Here a=p, b =7

If the lines are perpendicular

a + b = 0

(i.e.) p+7 = 0

p = -7

PART – B

1) Find the separate equations of the line $2x^2 - 7xy + 3y^2 = 0$. Also find the angle between them.

Solution:

$$2x^2 - 7xy + 3y^2 = 0$$

This is of the form $ax^2 + 2hxy + by^2 = 0$

(a=2, 2h = -7, h=-7/2, b=3)

$$2x^2 - 6xy - xy + 3y^2 = 0$$

$$2x(x - 3y) - y(x - 3y) = 0$$

$$(x - 3y)(2x - y) = 0$$

$\therefore$ The separate equations are

$x - 3y = 0$ and $2x - y = 0$

Let θ be the angle between the two straight lines

$$\therefore \tan\theta = \pm\frac{2\sqrt{h^2 - ab}}{a+b}$$

$$= \pm\frac{2\sqrt{(-7/2)^2 - (2)(3)}}{2+3}$$

$$= \pm\frac{2\sqrt{\frac{49}{4} - 6}}{5}$$

$$=\pm 2\frac{\sqrt{\frac{49-24}{4}}}{5}$$

$$=\pm 2\frac{\sqrt{\frac{25}{4}}}{5}$$

$$=\pm\frac{2x\frac{5}{2}}{5}$$

$\tan\theta = \pm 1$

$\tan\theta = \tan 45$, $\therefore \theta = 45°$

2) The slope of one of the lines $ax^2+2hxy+by^2=0$ is thrice that of the other. Show that $3h^2=4ab$

Solution:

$$ax^2+2hxy+by^2=0 \qquad (1)$$

Let $y=m_1x$ and $y=m_2x$ be the separate equations of equation (1)

$$m_1+m_2=\frac{-2h}{b} \qquad (2)$$

$$m_1m_2=\frac{a}{b} \qquad (3)$$

Slope of one of the line = thrice slope of the other line

(i.e.) $m_1 = 3m_2$

Equation (2) becomes

$$3m_2+m_2=-\frac{2h}{b}$$

$$4m_2=\frac{-2h}{b}$$

$$m_2=\frac{-2h}{4b}=\frac{-h}{2b}$$

Substitute $m_1=3m_2$ in equation (3)

$$3m_2.m_2 = \frac{a}{b}$$

$$(3m_2)^2 = \frac{a}{b}$$

$$3\left(\frac{-h}{2b}\right)^2 = \frac{a}{b}$$

$$3\left(\frac{-h^2}{4b^2}\right) = \frac{a}{b}$$

$$\frac{3h^2}{4b^2} = \frac{a}{b}$$

$$3h_2b = 4ab_2$$

$$(ie)3h^2 = 4ab$$

3.3 PAIR OF STRAIGHT LINES NOT PASSING THROUGH THE ORIGIN

Consider the second degree equation

$$(lx + my + n)(l'x + m'y + n') = 0 \qquad (1)$$

If (x_1, y_1) lies on $lx + my + n = 0$ (2)

then $lx_1 + my_1 + n = 0$. Hence (x_1,y_1) satisfies equation (1).

Similarly any point on $l'x + m'y + n' = 0$ (3)

also satisfies (1)

conversely, any point which satisfies (1) must be on any of the straight lines (2) and (3). Thus $(lx + my + n)\,(l'x + m'y + n') = 0$ represent a pair of lines.

Expanding equation (1) we get

$$ll'x^2 + lm'xy + lxn' + l'mxy + mm'y^2 + mn'y + l'xn + nm'y + nn' = 0$$

$$ll'x^2 + xy(lm' + l'm) + mm'y^2 + (ln' + l'n)x + (mn' + m'n)y + nn' = 0$$

Taking $a = ll'$ $\quad 2h = lm' + l'm$, $\quad b = mm'$

$2g = ln' + l'n$ $\quad 2f = mn' + m'n$ $\quad c = nn'$

We get $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

Condition for the second degree equation ax^2+2hxy+by^2+ 2hx+2fy+c to represent a pair of straight lines is abc+2fgh-af^2-bg^2-ch^2

(or)

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} 2a & 2h & 2g \\ 2h & 2b & 2f \\ 2g & 2f & 2c \end{vmatrix} = 0$$

1) Angle between pair of lines $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is

$$\tan\theta = \pm 2\frac{\sqrt{h^2 - ab}}{a + b}$$

2) The condition for the pair of lines to be parallel is $h^2 - ab = 0$

3) The condition for the pair of lines to be perpendicular is a + b=0.

3.3 WORKED EXAMPLES

PART - A

1) Find the combined equation of the lines whose separate equations are $2x - 3y + 2 = 0$ and $4x + y + 3 = 0$

Solution:

The two separate lines are $2x - 3y + 2 = 0$ and $4x + y + 3 = 0$

The combined equation of the given line is

$(2x - 3y + 2)\,(4x + y + 3) = 0$

$8x^2 + 2xy + 6x - 12xy - 3y^2 - 9y + 8x + 2y + 6 = 0$

(i.e.) $8x^2 - 10xy - 3y^2 + 14x - 7y + 6 = 0$

2) Show that the pair of lines given by $9x^2 + 24xy + 16y^2 + 21x$ $+28y + 6 = 0$ are parallel.

Solution:

This is of the form $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

Here a = 9, 2h = 24, b=16

h = 12

If the lines are parallel $h^2 - ab = 0$

(ie) $(12)^2 - (9)(16) = 0$

$= 144 - 144 = 0$

Hence the lines are parallel.

3) Show that the pair of lines given by $6x^2 + 3xy - 6y^2 - 8x$ $+5y - 3 = 0$ are perpendicular

Solution:

This is of the form $ax^2 + 2hxy + hy^2 + 2gx + 2fy + c = 0$

Here a = 6 b=-6

If the lines are perpendicular

a+b = 0

(i.e) 6 + (-6) = 0

Hence the pair of lines are perpendicular.

PART – B

1) Prove that equation $6x^2 + 13xy + 6y^2 + 8x + 7y + 2 = 0$ represents a pair of straight lines.

Solution:

Given equation

$$6x^2 + 13xy + 6y^2 + 8x + 7y + 2 = 0 \quad (1)$$

(i.e.) This is of the form

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + l = 0$$

Hence a = 6 b=6 c=2

2h=13, 2g = 8, 2f = 7

If the equation (1) represents a pair of straight lies then

$$\begin{vmatrix} 2a & 2h & 2g \\ 2h & 2b & 2f \\ 2g & 2f & 2c \end{vmatrix} = 0$$

$$\text{LHS} = \begin{vmatrix} 12 & 13 & 8 \\ 13 & 12 & 7 \\ 8 & 7 & 4 \end{vmatrix}$$

$$= 12(48 - 49) - 13(52 - 56) + 8(91 - 96)$$

$$= -12 + 52 - 40$$

$$= 0 = \text{RHS}$$

Hence the given equation represents a pair of straight lines.

2) Show that the equation $3x^2 + 7xy + 2y^2 + 5x + 5y + 2 = 0$ represents a pair of straight lines. Also find the separate equation of the lines.

Solution:

Given $3x^2 + 7xy + 2y^2 + 5x + 5y + 2 = 0$

This is of the form $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

a=3 b=2 c=2

2h=7 2g=5 2f=5

If the equation (1) represents a pair of straight lies then

$$\begin{vmatrix} 2a & 2h & 2g \\ 2h & 2b & 2f \\ 2g & 2f & 2c \end{vmatrix} = 0$$

$$\text{LHS} = \begin{vmatrix} 6 & 7 & 5 \\ 7 & 4 & 5 \\ 5 & 5 & 4 \end{vmatrix}$$

$$= 6(16 - 25) - 7(28 - 25) + 5(35 - 20)$$

$= -54 - 21 + 75$

$= 0 = RHS$

$\therefore$ The given equation represents a pair of straight lines.
Next we find separate lines.

Factorise the second degree terms

Let $3x^2 + 7xy + 2y^2 = 3x^2 + 6xy + xy + 2y^2$

$= 3x(x + 2y) + y(x + 2y)$

$= (x + 2y)(3x + y)$

$\therefore \; 3x^2 + 7xy + 2y^2 + 5x + 5y + 2 = (3x + y + l)(x + 2y + m)$ (say)

Equating the coefficient of x, l+3m=5 (2)

Equating the coefficient of y, 2l+m=5 (3)

Solving (2) and (3)

2l+6m = 10

2l+m = 5

5m=5

m=1

Sub in (2),

l + 3(1) = 5

l= 2

$\therefore$ The separate equation are 3x+y+2 =0 and x+2y+1=0

3) Find 'k' if $2x^2-7xy+3y^2+5x-5y+k = 0$ represents a pair of straight lines. Find the angle between them.

Solution :

$$2x^2 - 7xy + 3y^2 + 5x - 5y + k = 0$$

This is of the form $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

Hence a=2 b=3 c=k

2h=-7 2g=5 2f=-5

Since the given equation represents a pair of straight lines

$$\begin{vmatrix} 2a & 2h & 2g \\ 2h & 2b & 2f \\ 2g & 2f & 2c \end{vmatrix} = 0$$

$$\therefore \begin{vmatrix} 4 & -7 & 5 \\ -7 & 6 & -5 \\ 5 & -5 & 2k \end{vmatrix} = 0$$

$$4(12k-25)+7(-14k+25)+5(35-30)=0$$

48k – 100 – 98k + 175 + 25 = 0

-50k + 100 = 0

-50k = - 100

K = 2

It ‘θ’ is the angle between the given lines then

$$\tan\theta = \pm 2\frac{\sqrt{h^2-ab}}{a+b}$$

$$= \pm 2\frac{\sqrt{\left(\frac{-7}{2}\right)^2-(2)(3)}}{2+3}$$

$$= \pm 2\frac{\sqrt{\frac{49}{4}-6}}{5}$$

$$= \pm 2\frac{\sqrt{\frac{25}{4}}}{5}$$

$$= \pm 2\frac{\left(\frac{5}{2}\right)}{5}$$

tan θ = 1

tan θ = tan 45

$$\therefore \theta = \frac{\pi}{4}$$

4) Show that the pair straight lines $4x^2+4xy+y^2-6x-3y-4=0$ represents a pair of parallel straight lines and find the distance between them.

Solution:

Given: $4x^2+4xy+y^2-6x-3y-4=0$ (1)

This is of the form $ax^2+2hxy+by^2+2gx+2fy+c=0$

Here a = 4	2h = 4	h=2
b=1	2g=-6	g=-3
2f=-3	f=-3/2	c=-4

If the lines are parallel $h^2-ab=0$

(ie) $(2)^2$ (4) (1) = 0

4-4 = 0

$\therefore$The given equation (1) represents a pair of parallel straight lines.

To find the separate lines of (1)

Factorise $4x^2+4xy+4^2$

$$4x^2+2xy+2xy+y^2 = 2x(2x+y)+y(2x+y)$$
$$=(2x+y)(2x+y)$$
$$=(2x+y)^2$$

$4x^2+4xy+y^2-6x-3y-4=0$

$(2x+y)^2-3(2x+y)-4=0$

Let z = $2x+y$, then

(i.e.) $z^2-3z-4=0$

$(z+1)(z-4)=0$

$z+1=0$ and $z-4=0$

(i.e.) $2x+y+1=0$ and $2x+y-4=0$ are the separate equations

Distance between parallel lines

$2x+y-4=0$ and $2x+y+1=0$ is

$$\frac{|c_1 - c_2|}{\sqrt{a^2 + b^2}}$$

$$\frac{|-4 \ -1|}{\sqrt{(2)^2 + (1)^2}} \qquad = \frac{5}{\sqrt{5}} \quad = \sqrt{5}$$

EXERCISE

PART – A

1) Find the perpendicular distance from the point (3,-3) to the line 2x+4y+2=0
2) Find the perpendicular distance from the point (2,6) to the line x-4y+6=0
3) Find the length of the perpendicular to the line x-6y+5 = 0 from the origin.
4) Find the distance between the line 3x+3y+4=0 and 3x+3y-2 =0
5) Find the distance between the line x+2y-19=0 and x+2y-31 =0
6) Show that the lines 3x+2y-5=0 and 6x+4y-8 = 0 are parallel.
7) Show that the lines 2x-6y+6 = 0 and 4x-12y+7 = 0 are parallel.
8) Find the value of 'k' if the lines 7x-2y+13 = 0 and kx=3y+8 are parallel.
9) Find the value of 'p' if the lines 5x+3y = 6 and 3x+py = 7 are parallel.
10) Show that the lines 4x-3y=0 and 3x+4y+8 = 0 are perpendicular.
11) Show that the lines 4x-2y+6 = 0 and 2x+4y-4=0 are perpendicular
12) Find the value of 'p' if the lines 3x-py-4=0 and 2x+3y=7 are perpendicular.
13) Find the value of 'p' if the lines 2x-py+6=0 and 3x-2y+8 = 0 are perpendicular.

14) Find the slope of the line parallel to the line joining the points (3,4) and (-4,6)

15) Find the slope of the line perpendicular to the line joining the points (3,1) and (-4,3)

16) Show that the line joining the points (3,-5) and (-5,-4) is parallel to the line joining (7,10) and (15,9)

17) Show that the line joining the points (2,-2) and (3,0) is perpendicular to the line joining (2,2) and (4,1).

18) Find the equation of the line passing through (2,4) and parallel to the line $x+3y+7=0$

19) Find the equation of the line passing through (-2,5) and perpendicular to $5x-3y+8 = 0$

20) Write down the combined equation of the lines whose separate equation are

(i) $4x + 2y=0$ and $2x-y=0$

(ii) $3x + 2y=0$ and $2x - y=0$

(iii) $x + 2y = 0$ and $3x + 2y = 0$

(iv) $x + 2y = 0$ and $2x - y = 0$

21) Find the separate equation of each of the straight liens represents by

(i) $9x^2 - 16y^2 = 0$

(ii) $2x^2 - 5xy+2y^2 = 0$

(iii) $6x^2+xy-y^2=0$

(iv) $15x^2+17xy+2y^2=0$

22) Show that the two lines represented by $9x^2 + 6xy+y^2 = 0$ are parallel to each other.

23) Show that the equation $4x^2 - 12xy+9y^2 = 0$ represents a pair of parallel straight lines.

24) Find the values of p if the two straight lines represented by $20x^2+pxy+5y^2 = 0$ are parallel to each other.

25) Show that the pair of straight lines given by $2x^2-3xy-2y^2=0$ is perpendicular.

26) Find the value of 'p' so that the two straight lines represented by $px^2+6xy-y^2 = 0$ are perpendicular to each other.

27) Write down the combined equation of the lines whose separate equations are

(i) $x+2y = 0$ and $2x-y+1=0$

(ii) $x+2y=10$ and $2x-y-3=0$

(iii) $x+2y-1=0$ and $3x+2y+3=0$

28) Show that the equation $4x^2+4xy+y^2-6x-3y-4 =0$ represents two parallel lines.

29) Show that the equation $x^2+6xy+9y^2+4x+12y-5 = 0$ represents two parallel straight lines

30) Show that the equation $2x^2-3xy-2y^2+2x+y = 0$ represents a perpendicular pair of straight lines

PART – B

1) Show that the following equation represents a pair of straight line

(i) $9x^2-6xy+y^2+18x-6y+8 = 0$

(ii) $9x^2 + 24xy+16y^2+21x+28y+6 = 0$

(iii) $4x^2 + 4xy+y^2 – 6x-3y-4 = 0$

2) Find the angle between the pair of straight lines

3) Find the angle between the pair of lines given by $6x^2-13xy+5y^2= 0$. Find also the separate equations.

4) Find the angle between the pair of lines given by $9x^2+12xy+4y^2=0$. Find also the separate equations.

5) Find the angle between the pair of lines given by $3x^2-8xy+5y^2=0$. Find also the separate equation.

6) Find the separate equations of the pair of lines $3x^2-4xy+y^2=0$. Find also the angle between the lines.

7) If the slope of one of the lines of $ax^2+2hxy+by^2=0$ is twice the slope of the other, show that $8h^2= 9ab$

8) If the equation $ax^2+3xy-2y^2-5x+5y+c=0$ represents two lines perpendicular to each other, find the value of 'a' and 'c'.

9) Show that the equation $12x^2-10xy+2y^2+14x-5y+2=0$ represents a pair of lines. Also find the separate equations.

10) Show that the equation $12x^2+7xy-10y^2+13x+45y-35=0$ represents a pair of straight lines. Also find the separate equations.

11) Write down the separate equations of $3x^2 – 7xy – 6y^2 – 5x +26y – 8 = 0$. Also find the angle between the lines.

12) Show that the equation $9x^2+24xy+16y^2+21x+28y+6=0$ represents a pair of parallel straight lines. Find the separate equations and the distance between them.

13) Show that the equation $9x^2-6xy+y^2+18x-6y+8=0$ represents two parallel straight lines. Find the distance between them.

ANSWERS

PART – A

1. $\frac{-4}{\sqrt{20}}$	2. $\frac{-16}{\sqrt{17}}$	3. $\frac{5}{\sqrt{37}}$	4. $\frac{2}{\sqrt{18}}$
5. $\frac{12}{\sqrt{5}}$	8. $k=\frac{21}{2}$	9. $p=\frac{9}{5}$	5. P = 2
13. P = -3	14. $\frac{-2}{7}$	15. $\frac{7}{2}$	18. X+3y-14=0

19. $3x+5y-19 = 0$

20. (i) $8x^2 - 2y^2 = 0$

(ii) $6x^2 + xy - 2y^2 = 0$

(iii) $3x^2 + 8xy + 4y^2 = 0$

(iv) $2x^2 + 3xy - 2y^2 = 0$

21. (i) $(3x+4y)=0$, $3x-4y=0$ (ii) $x-2y=0, 2x-y=0$

(iii) $2x+y=0, 3x-y=0$ (iv) $x+y=0, 15x+2y=0$

24. $P = \pm 20$ 26. $P = 1$

27. (i) $2x^2+3xy-2y^2+x+2y=0$

(ii) $2x^2+3xy-2y^2-23x+4y+30=0$ (iii) $3x^2+8xy+4y^2+4y-3=0$

PART – B

(2) (i) $\theta = 60°$, (ii) $\theta = 90°$

(3) $\tan\theta = \frac{7}{11}$, $\theta = 32°, 28'$, 3x-5y=0, 2x-y =0

(4) $\theta=0$, 3x+2y=0, 3x+2y=0

(5) $\tan\theta = \frac{1}{4}$, 3x-5y=0, x-y=0

(6) 3x-y =0, x-y=0, $\tan\theta = \frac{1}{2}$

(8) a=2, c=-3

(9) 2x-y+2=0, 6x-2y+1=0

(10) 4x+5y-5=0

3x-2y+7 = 0

(11) 3x+2y-8=0

x-3y+1=0

$\tan\theta = \frac{11}{3}$

(12) 3x+4y+1=0

3x+4y+6=0

dist = 1

(13) $\frac{\sqrt{10}}{5}$

CIRCLES

Equation of circle – given centre and radius. General Equation of circle – finding center and radius. Simple problems.

Equation of circle through three non collinear points – concyclic points. Equation of circle on the line joining the points (x_1,y_1) and (x_2,y_2) as diameter. Simple problems.

Length of the tangent. Position of a point with respect to a circle. Equation of tangent (Derivation not required). Simple problems.

CIRCLES

Definition:

The locus or path of a point P(x,y) which is at a constant distance 'r' from a fixed point C(h,k) is called a circle.

The fixed point C(h,k) is called centre and the constant distance is called the radius of the circle.

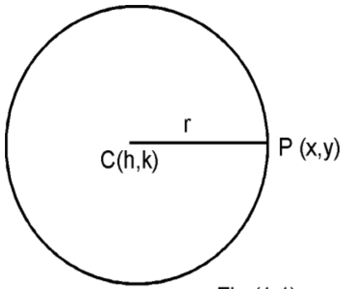


Fig (1.1)

Eqution of a circle with centre (h,k) and radius r:

Let the given centre and radius are C(h,k) and 'r' units. Let P(x,y) be any point on the circle. From Fig (1.1.) CP= r

ie $\sqrt{(x-h)^2+(y-k)^2}=r$ (using distance formula)

$$(x-h)^2+(y-k)^2=r^2 \qquad (1)$$

Note: When centre is at the origin (0,0) the equation (1) becomes $x^2+y^2=r^2$. i.e. the equation of the circle with centre at the origin and radius 'r' units is $x^2+y^2=r^2$.

General Equation of the circle.

The general equation of the circle is

$$x^2+y^2+2gx+2fy+c=0 \qquad (2)$$

Equation (2) can be re written as

$$x^2+2gx+g^2+y^2+2fy+f^2=g^2+f^2-c \quad \text{(or)}$$

$$(x+g)^2+(y+f)^2=g^2+f^2-c$$

$$\Big[\qquad {}^2 \qquad {}^2 x-\left(\sqrt{g^2+f^2-c}\right)^2 \qquad (3)$$

Equation (3) is in the form of equation (1)

$\therefore$ The eqution (2) represents a circle with centre (-g,-f) and radius $\sqrt{g^2+f^2-c}$.

Note:

(i) Coefficient of x^2 = coefficient of y^2

(ii) Centre of the circle = $\left[-\frac{1}{2}\text{coefficient of } x, -\frac{1}{2}\text{coefficient of } y\right]$

(iii) Radius = $\sqrt{g^2+f^2-c}$

WORKED EXAMPLES

PART - A

1) Find the equation of the circle whose centre is (2,-1) and radius 3 units.

Solution:

Equation of the circle with centre (h,k) and radius 'r' is

$(x-h)^2+(y-k)^2=r^2$ (h,k) =(2,-1) r=3

$\therefore (x-2)^2+(y+1)^2=3^2$ $x^2+y^2-4x+2y-4=0$

2) Find the centre and radius of the circle $x^2+y^2-6x+4y+2=0$

Solution:

Here 2g = -6 and 2f = 4

g = -3 f = 2

Centre is (-g,-f)

Centre (3,-2)

$\therefore$ r = $\sqrt{(-3)^2+2^2-2}$

$\therefore$ r = $\sqrt{11}$units

PART – B

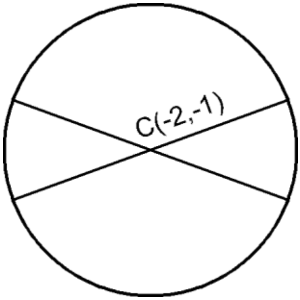
1) If 3x-y+5 = 0 and 4x+7y+15=0 are the equations of two diameters of a circle of radius 4 units write down the equation of the circle.

Solution:

Given diameters are

3x-y+5=0 (1)

4x+7y+15 = 0 (2)



Solving (1) and (2) we get x=-2 and y = -1

$\therefore$ centre (-2,-1) given radius r = 4

$\therefore$ Equation of the circle is

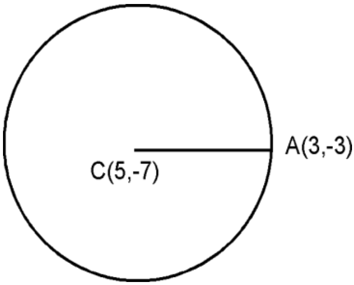
$(x-h)^2+(y-k)^2=r^2$

$(x+2)^2+(y+1)^2=4^2$

i.e. $x^2+y^2+4x+2y-11=0$

2) Find the equation of the circle whose centre is (5,-7) and passing through the point (3,-3)

Solution:



Let the centre and a point on the circle be C(5,-7) and A (3,-3)

$\therefore \text{Radius} = CA = \sqrt{(5-3)^2+(-7+3)^2}$

$r=\sqrt{4+16}$ $\qquad$ $r=\sqrt{20}$

Equation of circle is $(x-h)^2+(y-k)^2=r^2$

(h,k) = (5,-7) $\quad r=\sqrt{20} \quad r^2 = 20$

$\therefore (x-5)^2+(y+7)^2=20$

$x^2-10x+25+y^2+14y+49-20=0$

i.e. $x^2+y^2-10x+14y+54=0$

CONCYCLIC POINTS

If four or more points lie on the same circle the points are called concylic points.

1.2.1 Equation of circle with end points of a diameter

Let $A(x_1,y_1)$ and $B(x_2,y_2)$ be given two end points of a diameter. Let P (x,y) be any point on the circle.

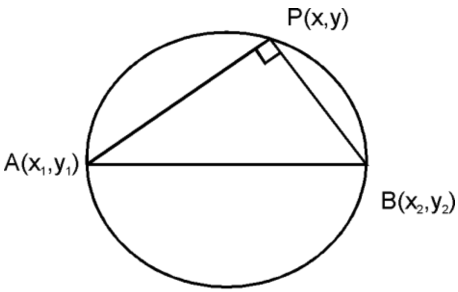


Fig (1.2)

i.e $\angle APB=90^\circ$ (Angle in a Semi–circle is 90°)

$AP \perp PB$

$\therefore$(slope of AP) (slope of PB) = -1

$$\left(\frac{y-y_1}{x-x_1}\right)\left(\frac{y-y_2}{x-x_2}\right)=-1$$

ie. $(y-y_1)(y-y_2) = -(x-x_1)(x-x_2)$

ie $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$

is the required equation of the circle.

WORKED EXAMPLES

PART – A

1) Find the equation of the circle joining the points (1,-1) and (-2,3) as diameter

Solution:

Equation of the circle is

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$$

$$(x_1 y_1)=(1,-1) \qquad (x_2, y_2)=(-2,3)$$

$$\therefore (x-1)(x+2)+(y+1)(y-3)=0$$

$$x^2+y^2+x-2y-5=0$$

2) Find the equation of the circle joining the points (a,0) and (0,b) as diameter

Solution:

Equation of the circle is

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$$

$$(x_1, y_1)=(a,0) \text{ and } (x_2, y_2)=(0,b)$$

$$\therefore (x-a)(x-0)+(y-0)(y-b)=0$$

$$x^2+y^2-ax-by=0$$

PART – B

1) Find the equation of the circle passing through the points (1,1), (1,0) and (0,1)

Solution:

Let the equation of the circle be

$$x^2+y^2+2gx+2fy+c=0 \qquad (1)$$

(1,1) lies on (1) i.e

$$1^2+1^2+2g(1)+2f(1)+c=0$$

$$2g+2f+c = -2 \qquad (2)$$

(1,0) lies on (1)

i.e $1^2+0^2+2g(1)+2f(0)+c=0$

$$2g+c = -1 \qquad (3)$$

(0,1) lies on (1)

i.e $0^2+1^2+2g(0)+2f(1)+c=0$

$$2f+c= -1 \qquad (4)$$

$$\begin{array}{l} 2g+2f+c = -2 \\ 2g+0+c=-1 \\ \hline 0 + 2f+0=-1 \\ \hline \end{array}$$

(2) – (3) →

$$2f=-1 \quad \therefore f= -\frac{1}{2}$$

Substitute $f=-\frac{1}{2}$ in (4)

$$2\left(-\frac{1}{2}\right)+c=-1$$

$$-1+c=-1 \quad \therefore c= 0$$

Substitute c= 0 in (3)

$$2g + 0 = -1$$

$$g = -\frac{1}{2}$$

$\therefore$ Equation of the circle is

$$x^2 + y^2 + 2\left(\frac{-1}{2}\right)x + 2\left(\frac{-1}{2}\right)y + 0 = 0$$

ie. $x^2 + y^2 - x - y = 0$

2) Find the equation of the circle passing through the points (0,1) , (4,3) and having its centre on the line 4x-5y-5=0

Solution:

Let the Equation of the circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad (1)$$

(0,1) lies on (1)

i.e $0^2 + 1^2 + 2g(0) + 2f(1) + c = 0$

$$2f+c = -1 \qquad (2)$$

(4,3) lies on (1)

i.e. $4^2 + 3^2 + 2g(4) + 2f(3) + c = 0$

$$8g + 6f + c = -25 \qquad (3)$$

Centre (-g,-f) lies on the line $4x - 5y - 5 = 0$

i.e $4(-g) - 5(-f) - 5 = 0 \qquad (4)$

$$-4g + 5f = 5$$

$$2f + c = -1$$

$$8g + 6f + c = -25$$

$(2) - (3) \rightarrow \quad \overline{-8g - 4f + 0 = 24}$

$$4g + 2f = -12 \qquad (5)$$

(4) + (5) gives;

$$\begin{aligned} -4g + 5f &= 5 \\ 4g + 2f &= -12 \\ \hline 0 + 7f &= -7 \\ 7f &= -7 \end{aligned} \qquad \therefore f = -1$$

Substituting f=-1 in (5)

4g-2 (-1) = -12

4g+2 = -12   4g = -10 $\therefore g = \frac{-10}{4}$ i.e., $g = \frac{-5}{2}$

Substituting $g = \frac{-5}{2}$, f=-1 in (3)

$$8\left(\frac{-5}{2}\right) + 6(-1) + c = -25$$

$-20 - 6 + c = -25$ $c = -25 + 26$ c = 1

$\therefore$ Equation of the required circle is

$$x^2 + y^2 + 2\left(\frac{-5}{2}\right)x + 2(-1)y + 1 = 0$$

$$x^2 + y^2 - 5x - 2y + 1 = 0$$

3) Show that the points (4,1), (6,5) (2,7) and (0,3) are concyclic

Solution:

Let the equation of the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$ (1)

(4,1) lies on the circle (1)

$$4^2 + 1^2 + 2g(4) + 2f(1) + c = 0$$

$$8g + 2f + c = -17 \qquad (2)$$

(6,5) lies on the circle (1)

$$\therefore\ 6^2+5^2+2g(6)+2f(5)+c=0$$

12g+10f+c = -61 (3)

(2,7) lies on the circle (1)

$$\therefore\ 2^2+7^2+2g(2)+2f(7)+c=0$$

4g+14f+c = -53 (4)

(3) – (2) → 4g +8f = -44 (5)

(3)-(4) → 8g – 4f = -8 (6)

Solving (5) and (6) we get g=-3 and f=-4

Substituting g = -3, f=-4 in (2)

8(-3) + 2(-4)+c = -17

C = -17 +32 = 15

∴ Equation of circle passing through the three points (4,1) (6,5) and (2,7) is

$$x^2+y^2+2(-3)x+2(-4)y+15=0$$

$$x^2+y^2-6x-8y+15=0 \qquad (7)$$

Substituting the fourth point (0,3) in (7)

$$0^2+3^2-6(0)-8(3)+15=0$$

$$9-24+15=0 \qquad \therefore\ 24-24=0$$

∴ (0,3) also lies on (7)

Hence the given four points are concyclic.

Length of the Tangent to a circle from a point (x_1,y_1)

Let the equation of the circle be $x^2+y^2+2gx+2fy+c=0$ and the point A(x_1,y_1) lies out side the circle.

We know that the centre is C(-g,-f) and radius $BC = r = \sqrt{g^2 + f^2 - c}$

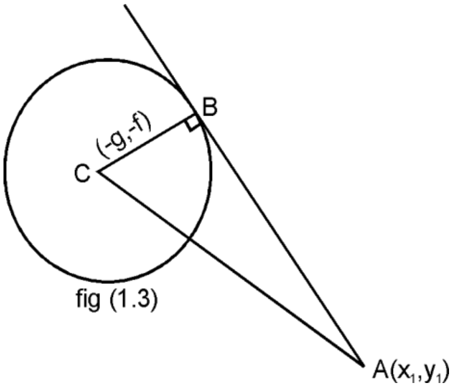


fig (1.3)

From the fig (1.3)

Δ ABC is a right angled triangle.

$\therefore\ AB^2 + BC^2 = AC^2$

$AB^2 = AC^2 - BC^2$

$AB^2 = (x_1 + g)^2 + (y_1 + f)^2 - (g^2 + f^2 - c)$

$AB^2 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + C$

$AB = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$

Which is the length of the tangent from the point (x_1,y_1) to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$

Note:

(i) If AB > 0, $A(x_1,y_1)$ lies out side the circle.

(ii) If AB < 0, $A(x_1,y_1)$ lies inside the circle.

(iii) If AB = 0, A (x_1,y_1) lies on the circle.

Equation of the Tangent to a circle at the point (x_1,y_1) onthe circle (Results only):

Given equation of circle is $x^2 + y^2 + 2gx + 2fy + c = 0$

The point A (x_1,y_1) lies on the circle.

i.e $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0$

From fig (1.4) AT is the tangent at A. We know that centre is C(-g,-f).

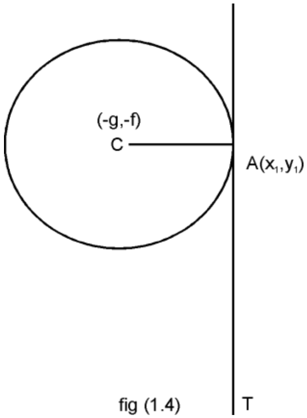


fig (1.4)

Slope of AC = $\dfrac{y_1 + f}{x_1 + g}$

Since AC is perpendicular to AT

Slope of AT = m = $-\dfrac{(x_1 + g)}{(y_1 + f)}$

$\therefore$ Equation of the tangent AT at A(x_1,y_1) is

$$y - y_1 = m(x - x_1)$$

$$y - y_1 = -\left(\frac{x_1 + g}{y_1 + f}\right)(x - x_1)$$ on simplification, we get

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

Note: The equation of tangent to the circle $x^2 + y^2 = r^2$ at (x_1,y_1) is obtained by substituting g=0, f=0 and c=-r^2 in the above equation to tangent

$$\therefore\ xx_1 + yy_1 + 0(x + x_1) + 0(y + y_1) - r^2 = 0$$

ie $xx_1 + yy_1 = r^2$

1.3.2 RESULTS

1) Equation of the tangent to a circle at a point (x_1,y_1) is
$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

2) Length of the tangent from the point (x_1,y_1) to the circle
$x^2 + y^2 + 2gx + 2fy + c = 0$ is $\sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$

Note:

(1) the equation of the tangent to the circle $x^2+y^2 = r^2$ at (x_1,y_1) is $xx_1 + yy_1 = r^2$

(2) If $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c > 0$ then the point (x_1,y_1) lies outside the circle.

(3) If $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0$ then the point (x_1,y_1) lies on the circle.

(4) If $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c < 0$, then the point (x_1,y_1) lies inside the circle.

WORKED EXAMPLES

PART – A

1) Find the length of the tangent from (2,3) to the circle
$x^2 + y^2 - 2x + 4y + 1 = 0$

Solution:

Length of the tangent = $\sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$

$= \sqrt{2^2 + 3^2 - 2(2) + 4(3) + 1}$

$= \sqrt{4 + 9 - 4 + 12 + 1}$

$= \sqrt{22}$ units

2) Show that the point (9,2) lies on the circle
$x^2 + y^2 - 6x - 10y - 11 = 0$

Solution:

Substitute (9,2) on the circle $x^2 + y^2 - 6x - 10y - 11 = 0$

$9^2 + 2^2 - 6(9) - 10(2) - 11 = 0$

81 + 4-54-20-11=0. $\therefore$ 85-85 = 0

$\therefore$ the point (9,2) lies on the circle.

3) Find the equation of the tangent at (-4,3) to the circle $x^2 + y^2 = 25$

Solution:

Equation of tangent is $xx_1+yy_1=r^2$

$\therefore$ x(-4) + y(3) = 25

-4x+3y =25

4x-3y+25 = 0

PART – B

1) Find the equation of the tangent at (4,1) to the circle $x^2 + y^2 - 8x - 6y + 21 = 0$

Solution:

Equation of the tangent at the point (x_1,y_1) is
$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

Given $x^2 + y^2 - 8x - 6y + 21 = 0$

2g = -8, 2f = -6 c = 21 (x_1,y_1) = (4,1)
g = -4 f =-3

$x(4) + y(1) + (-4)[x + 4] + (-3)[y + 1] + (21) = 0$

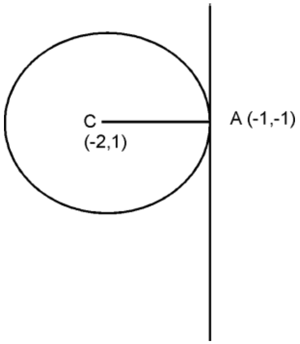
$4(x) + y - 4x - 16 - 3y - 3 + 21 = 0$

-2y+2 = 0 2y-2 = 0 $\therefore$y-1 = 0

Equation of the tangent is y -1 = 0

2) Find the equation of the tangent to the circle $(x+2)^2 + (y-1)^2 = 5$ at (-1,-1).

Solution:



Equation of the circle in $(x+2)^2 + (y-1)^2 = 5$

i.e $x^2 + y^2 + 4x - 2y = 0$

Here $2g=4$ $\quad$ $2f = -2$

$g=2$ $\quad$ $f=-1$ $\quad$ $c= 0$

$\therefore$ Equation of the tangent is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

i.e., $x(-1) + y(-1) + 2(x - 1) - 1(y - 1) = 0$

$-x - y + 2x - 2 - y + 1 = 0$ $\quad$ $x-2y-1 = 0$

EXERCISE
PART – A

1. Find the equation of the circle whose centre and radius are given as

(i) (3,2); 4 units $\quad$ (ii) (-5,7) , 3 units

(iii) (-5,-4); 5 units $\quad$ (iv) (6,-2), 10 units

2. Find the centre and radius of the following circles:

(i) $x^2 + y^2 - 12x - 8y + 2 = 0$ (ii) $x^2 + y^2 + 7x + 5y - 1 = 0$

(iii) $2x^2 + 2y^2 - 6x + 12y - 4 = 0$ (iv) $x^2 + y^2 = 100$

3. Write down the equation of circle whose centre is (h,k) and radius 'r' units.

4. Write down the centre and radius of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$

5. Find the centre and radius of the circle $(x-2)^2 + (y+3)^2 = 16$

6. Find the equation of the circle described on line joining the following points as diameter:

(i) (3,5) and (2,7) (ii) (-1,0) and (0,-3)

(iii) (0,0) and (4,4) (iv) (-6,-2) and (-4,-8)

7. Write down the equation of the circle whose end points of the diameter are (x_1,y_1) and (x_2,y_2)

8. Write down the expression to find the length of the tangent to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ from the point (x_1,y_1)

9. Write down the equation of the tangent to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at the point (x_1,y_1).

10. Find the length of the tangent from the point (2,1) to the circle $x^2 + y^2 + 2x + 4y + 3 = 0$

11. Show that the point (-3,-4) lies inside the circle $x^2 + y^2 + 2x + y - 25 = 0$

12. Show that the point (-1,-7) lies on the circle $x^2 + y^2 + 15x + 2y - 21 = 0$

PART – B

1) x+2y=1 and 3x-4y=3 are two diameters of a circle of radius 5 units. Find the equation of the circle.

2) Find the equation of the circle two of its diameters are 3x+4y=2 and x-y =3 and passing through (5,-1)

3) Find the equation of the circle passing through the points (5,2), (2,1), (1,4).

4) Find the equation of the circle passing through the points (6,0) and (-1,-1) and having its centre on x+2y+5=0

5) Prove that the points (3,4), (0,5),(-3,-4) and (-5,0) are concylic

6) Find the equation of the tangent at (2,4) to the circle $x^2 + y^2 + 2x - 4y - 8 = 0$

7) Find the equation of the tangent at (-7,-11) to the circle $x^2 + y^2 = 500$

8) Show that the point (1,-4) lies on the circle $x^2 + y^2 - 12x + 4y + 11 = 0$ Also find the equation of the tangent at (1,-4).

ANSWER

PART – A

(1) (i) $x^2 + y^2 - 6x - 4y - 3 = 0$

(ii) $x^2 + y^2 + 10x - 14y + 65 = 0$

(iii) $x^2 + y^2 + 10x + 8y + 16 = 0$

(iv) $x^2 + y^2 - 12x + 4y - 60 = 0$

(2) (i) (6,4), $\sqrt{50}$

(ii) $\left(\frac{-7}{2}, \frac{-5}{2}\right)$; $\frac{\sqrt{78}}{2}$

(iii) $\left(\frac{3}{2}, -3\right)$; $\frac{\sqrt{53}}{2}$

(iv) (0,0); 10.

(5) (2,-3), 4

(6) (i) $x^2 + y^2 - 5x - 12y + 41 = 0$

(ii) $x^2 + y^2 + x + 3y = 0$

(iii) $x^2 + y^2 - 4x - 4y = 0$

(iv) $x^2 + y^2 + 10x + 10y + 40 = 0$

(10) 4 units

(11) $\sqrt{-10}$