

LEARNING MATERIAL ON DERIVATIVES

**SEMESTER : II
DEPARTMENT : MATHEMATICS AND SCIENCE
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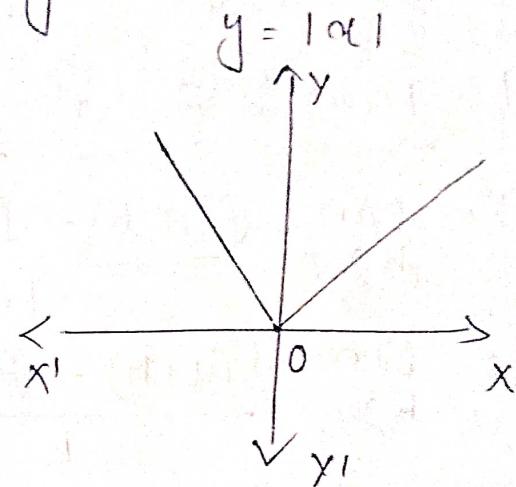
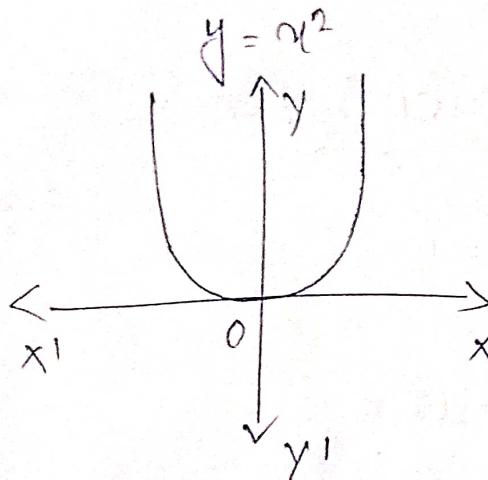


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DERIVATIVES

Graphical understanding :-



continuous and smooth

continuous and edged

* A function is differentiable if its graph is continuous and smooth.

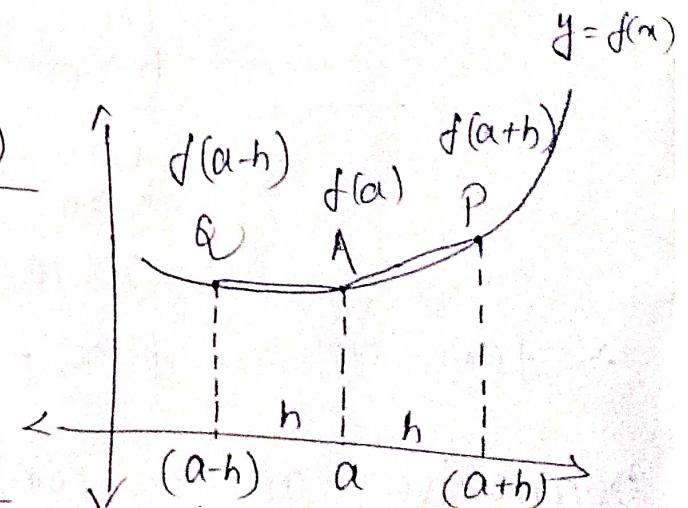
• $f(x) = x^2 \Rightarrow f'(x)$ or $\frac{d}{dx} f(x) = 2x$

• $f(x) = |x| \Rightarrow f'(x)$ or $\frac{d}{dx} f(x)$ — Does not exist.

Concept of Tangent :-

Slope of PA = $\frac{f(a+h) - f(a)}{(a+h) - a}$
 $= \frac{f(a+h) - f(a)}{h}$

Slope of QA = $\frac{f(a) - f(a-h)}{a - (a-h)}$
 $= \frac{f(a) - f(a-h)}{-h} = \frac{f(a-h) - f(a)}{h}$



Slope of tangent at A = $\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = \text{RHD}$

Slope of tangent at A = $\lim_{h \rightarrow 0} \frac{f(a-h) - f(a)}{-h} = \text{LHD}$

For differentiability, LHD = RHD = Finite (l)

* If LHD = RHD = Finite, then $f(a)$ is differentiable at $a=a$.

First Principle of Differentiation :-

$$LHD = \lim_{h \rightarrow 0} \frac{f(a-h) - f(a)}{-h} = l$$

$$RHD = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = l$$

$$\Rightarrow \boxed{\frac{d}{dx} f(x) \text{ or } f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = l}$$

e.g:- $f(x) = x^2$

$$\frac{d}{dx} f(x) = \frac{d}{dx} x^2 = \lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h \rightarrow 0} \frac{x^2 + 2hx + h^2 - x^2}{h}$$

$$= \lim_{h \rightarrow 0} \frac{h(2x+h)}{h}$$

$$= \lim_{h \rightarrow 0} 2x+h$$

$$\Rightarrow 2x+0 = 2x$$

$$\Rightarrow f(x) = x^2 \Rightarrow f'(x) = \frac{d}{dx} f(x) = 2x$$

Derivative at a Point :-

A function is said to be derivable or differentiable at $x \in (a,b)$, if $\lim_{h \rightarrow 0} \frac{f(x+h)-f(x)}{h}$ exists and this limit is denoted by $f'(x)$ or $\frac{d}{dx} f(x)$ or $Df(x)$.

$\frac{d}{dx}$ or $D \rightarrow$ Differential operator

The derivative of $y = f(x)$ is also denoted by $\frac{dy}{dx}$ or y' or Dy .

The process of finding derivatives is called differentiation or derivation.

Algebra of Derivatives :-

$$* \frac{d}{dx} (f(x) + g(x)) = \frac{d}{dx} f(x) + \frac{d}{dx} g(x)$$

$$* \frac{d}{dx} (f(x) - g(x)) = \frac{d}{dx} f(x) - \frac{d}{dx} g(x)$$

$$* \frac{d}{dx} (k f(x)) = k \frac{d}{dx} f(x)$$

$$* \frac{d}{dx} (f(x) \cdot g(x)) = g(x) \frac{d}{dx} f(x) + f(x) \frac{d}{dx} g(x)$$

$$* \frac{d}{dx} \left(\frac{f(x)}{g(x)} \right) = \frac{g(x) \frac{d}{dx} f(x) - f(x) \frac{d}{dx} g(x)}{(g(x))^2}$$

Derivative of Standard Functions :-

1. $\frac{d}{dx} (a) = 0$, where $a \in \mathbb{R}$
2. $\frac{d}{dx} (x^n) = n x^{n-1}$, where $n \in \mathbb{R}$ and $n \neq 0$
3. $\frac{d}{dx} a^x = a^x \ln a$
4. $\frac{d}{dx} e^x = e^x$
5. $\frac{d}{dx} |\ln x| = \frac{1}{x}$
6. $\frac{d}{dx} \log_a x = \frac{1}{x \ln a}$
7. $\frac{d}{dx} \sin x = \cos x$

$$8. \frac{d}{dx} \cos x = -\sin x$$

$$9. \frac{d}{dx} \tan x = \sec^2 x$$

$$10. \frac{d}{dx} \cot x = -\operatorname{cosec}^2 x$$

$$11. \frac{d}{dx} \sec x = \sec x \cdot \tan x$$

$$12. \frac{d}{dx} \operatorname{cosec} x = -\operatorname{cosec} x \cdot \cot x$$

$$13. \frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}, |x| < 1$$

$$14. \frac{d}{dx} \cos^{-1} x = \frac{-1}{\sqrt{1-x^2}}, |x| < 1$$

$$15. \frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}, x \in \mathbb{R}$$

$$16. \frac{d}{dx} \cot^{-1} x = \frac{-1}{1+x^2}, x \in \mathbb{R}$$

$$17. \frac{d}{dx} \sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}}, |x| > 1$$

$$18. \frac{d}{dx} \operatorname{cosec}^{-1} x = \frac{-1}{|x|\sqrt{x^2-1}}, |x| > 1$$

E.g. • $y = \sin x + e^x \cos x$

$$\Rightarrow y' = \frac{d}{dx} (\sin x + e^x \cos x)$$

$$= \frac{d}{dx} \sin x + \frac{d}{dx} e^x \cos x$$

$$= \cos x + \cos x \frac{d}{dx} e^x + e^x \frac{d}{dx} \cos x$$

$$= \cos x + e^x \cos x - e^x \sin x$$

$$= \cos x + e^x (\cos x - \sin x)$$

• $y = ax^2 + bx + c$

$$\Rightarrow y' = \frac{d}{dx} (ax^2 + bx + c)$$

$$\begin{aligned}
 &= \frac{d}{dx} ax^2 + \frac{d}{dx} b\sqrt{x} + \frac{d}{dx} cx \\
 &= a \frac{d}{dx} x^2 + b \frac{d}{dx} x^{1/2} + c \frac{d}{dx} x \\
 &= a \cdot 2x + b \cdot \frac{1}{2} x^{-1/2} + c \\
 &= 2ax + \frac{1}{2} bx^{-1/2} + c
 \end{aligned}$$

$$\bullet y = e^x \sin x$$

$$\begin{aligned}
 \Rightarrow y' &= \frac{d}{dx} (e^x \sin x) \\
 &= e^x \sin x \frac{d}{dx} x + x \sin x \frac{d}{dx} e^x + xe^x \frac{d}{dx} \sin x \\
 &= e^x \sin x + x \sin x \cdot e^x + xe^x \cos x \\
 &\Rightarrow e^x (\sin x + x \sin x + x \cos x)
 \end{aligned}$$

$$\Rightarrow y' = e^x (\sin x + x \sin x + x \cos x)$$

$$\bullet y = \frac{1+x}{1-x}$$

$$\begin{aligned}
 \Rightarrow y' &= \frac{(1-x) \frac{d}{dx} (1+x) - (1+x) \frac{d}{dx} (1-x)}{(1-x)^2} \\
 &= \frac{(1-x)(0+1) - (1+x)(0-1)}{(1-x)^2} \\
 &= \frac{1-x + 1+x}{(1-x)^2} = \frac{2}{(1-x)^2}
 \end{aligned}$$

$$\bullet y = \frac{x^4 + x^2 - 1}{x^2 - x + 1}$$

$$\begin{aligned}
 \Rightarrow y' &= \frac{(x^2 - x + 1) \frac{d}{dx} (x^4 + x^2 - 1) - (x^4 + x^2 - 1) \frac{d}{dx} (x^2 - x + 1)}{(x^2 - x + 1)^2} \\
 &= \frac{(x^2 - x + 1)(4x^3 + 2x) - (x^4 + x^2 - 1)(2x - 1)}{(x^2 - x + 1)^2}
 \end{aligned}$$

$$= 4x^5 + 2x^3 - 4x^4 - 2x^2 + 4x^3 + 2x - 2x^5 + x^4 - 2x^3$$

$$+ x^2 + 2x - 1$$

$$(x^2 - x + 1)^2$$

$$= \frac{2x^5 - 3x^4 + 4x^3 - x^2 + 4x - 1}{(x^2 - x + 1)^2}$$

$$= \frac{(2x+1)(x^4 - 2x^3 + 3x^2 - 2x + 1)}{(x^2 - x + 1)^2}$$

$$= \frac{(2x+1)(x^2 - x + 1)^2}{(x^2 - x + 1)^2} = 2x+1$$

OR, $y = \frac{x^4 + x^2 - 1}{x^2 - x + 1}$

$$\Rightarrow y = \frac{(x^4 + 2x^2 + 1) - x^2}{(x^2 - x + 1)}$$

$$= \frac{(x^2 + 1)^2 - x^2}{(x^2 - x + 1)}$$

$$= \frac{(x^2 + 1 - x)(x^2 + 1 + x)}{(x^2 - x + 1)}$$

$$\Rightarrow y = (x^2 + x + 1)$$

$$\Rightarrow y' = \frac{d}{dx}(x^2 + x + 1)$$

$$= \frac{d}{dx}x^2 + \frac{d}{dx}x + \frac{d}{dx}1$$

$$\Rightarrow 2x + 1 + 0$$

$$\Rightarrow y' = 2x + 1$$

* Simplification \rightarrow Differentiation.

Derivative of a composite function :-

[The chain Rule]

Let $y = f(u)$ be a differentiable function of u and $u = g(x)$ be a differentiable function of x .

i.e $y = f(g(x))$ or $y = fog(x)$ is a composite function of x .

Then,
$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

i.e $\frac{d}{dx}(f(g(x))) = \frac{d}{dg(x)}(f(g(x))). \frac{d}{dx}g(x)$

$$\Rightarrow \frac{d}{dx}(f(g(x))) = f'(g(x)). g'(x)$$

This is called the chain rule of differentiation.

e.g. $y = (x^2 + 2x - 1)^5$

Here $y = f(u)$ s.t $u = x^2 + 2x - 1$

$$\therefore y = u^5 \Rightarrow \frac{dy}{du} = \frac{d}{du}(u^5) = 5u^4 = 5(x^2 + 2x - 1)^4$$

$$\frac{du}{dx} = \frac{d}{dx}(x^2 + 2x - 1) = 2x + 2$$

$$\begin{aligned}\therefore \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} = 5(x^2 + 2x - 1)^4 \cdot (2x + 2) \\ &= 10(x+1)(x^2 + 2x - 1)^4\end{aligned}$$

$y = \cot^3 x$

$$\Rightarrow y' = 3 \cot^2 x \cdot (-\operatorname{cosec}^2 x)$$

$$\Rightarrow -3 \cot^2 x \cdot \operatorname{cosec}^2 x.$$

Note :-

- * $\frac{d}{dx} (ax+b)^d = d(ax+b)^{d-1} \cdot a, \quad d \in \mathbb{R}$
- * $\frac{d}{dx} a^{bx} = a^{bx} \ln a \cdot b = ba^{bx} \ln a$
- * $\frac{d}{dx} e^{ax} = e^{ax} \cdot a = ae^{ax}$
- * $\frac{d}{dx} \sin(ax+b) = a \cos(ax+b)$
- * $\frac{d}{dx} \cos(ax+b) = -a \sin(ax+b)$
- * $\frac{d}{dx} \ln(ax+b) = \frac{a}{ax+b}$
- * $\frac{d}{dx} \tan(ax+b) = a \sec^2(ax+b)$
- * $\frac{d}{dx} \sec(ax+b) = a \sec(ax+b) \tan(ax+b)$
- * $\frac{d}{dx} \cot(ax+b) = -a \operatorname{cosec}^2(ax+b)$
- * $\frac{d}{dx} \operatorname{cosec}(ax+b) = -a \operatorname{cosec}(ax+b) \cot(ax+b)$

e.g:-

$$\begin{aligned}
 & \frac{d}{dx} \cos(\ln x)^2 \\
 &= -\sin(\ln x)^2 \cdot \frac{d}{dx} (\ln x)^2 \\
 &= -\sin(\ln x)^2 \cdot 2 \ln x \frac{d}{dx} \ln x \\
 &= -\sin(\ln x)^2 \cdot 2 \ln x \cdot \frac{1}{x} \\
 &= -\frac{2}{x} (\ln x) \sin(\ln x)^2.
 \end{aligned}$$

$$\begin{aligned}
 & \frac{d}{d\theta} (\sec(\tan \theta)) \\
 &= \sec(\tan \theta) \cdot \tan(\tan \theta) \frac{d}{d\theta} (\tan \theta) \\
 &= \sec(\tan \theta) \cdot \tan(\tan \theta) \cdot \sec^2 \theta
 \end{aligned}$$

Derivatives of Inverse functions :-

Let f be a differentiable function of α which admits of an inverse function f^{-1} , then

$$\frac{d f^{-1}}{dy} = \frac{1}{\left(\frac{df}{d\alpha}\right)} \text{ Provided } \frac{df}{d\alpha} \neq 0$$

OR, $\frac{d\alpha}{dy} = \frac{1}{\left(\frac{dy}{d\alpha}\right)}$, Provided $\frac{dy}{d\alpha} \neq 0$.

E.g. - $\alpha = \frac{\sqrt{y}}{\sqrt{y} + 1}$ then $\frac{dy}{d\alpha} = ?$

Ans:- $\alpha = \frac{\sqrt{y}}{\sqrt{y} + 1} = 1 - \frac{1}{\sqrt{y} + 1}$

$$\begin{aligned}
 \therefore \frac{d\alpha}{dy} &= \frac{d}{dy} \left(1 - \frac{1}{\sqrt{y} + 1} \right) = 0 - \frac{d}{dy} \left(\frac{1}{\sqrt{y} + 1} \right) = -\frac{d}{dy} (\sqrt{y} + 1)^{-1} \\
 &= -(-1)(\sqrt{y} + 1)^{-2} \frac{d}{dy} (\sqrt{y} + 1) \\
 &\stackrel{2}{=} (\sqrt{y} + 1)^{-2} \left(\frac{1}{2} y^{-1/2} + 0 \right) \\
 &= \frac{y^{-1/2}}{2(\sqrt{y} + 1)^2} = \frac{1}{2y^{1/2}(\sqrt{y} + 1)^2}
 \end{aligned}$$

$$\Rightarrow \frac{dy}{d\alpha} = 2y^{1/2}(\sqrt{y} + 1)^2 = 2\sqrt{y}(\sqrt{y} + 1)^2$$

Derivatives of Inverse Trigonometric functions :-

For $y = \sin^{-1}\alpha$, $|\alpha| < 1$

$$\Rightarrow \sin y = \alpha$$

Differentiating both sides, we have -

$$\Rightarrow \frac{d\alpha}{dy} = \cos y$$

$$\Rightarrow dy/d\alpha = 1/\cos y$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$$

$$\therefore \boxed{\frac{d}{dx}(\sin^{-1} \alpha) = \frac{1}{\sqrt{1-\alpha^2}}, |\alpha| < 1}$$

similarly we can prove the other formulas.

e.g. • $y = \tan^{-1} (\sin^2 x)$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx} (\tan^{-1} (\sin^2 x))$$

$$= \frac{1}{1+(\sin^2 x)^2} \cdot \frac{d}{dx} (\sin^2 x)$$

$$= \frac{1}{1+\sin^4 x} \cdot 2 \sin x \cdot \frac{d}{dx} \sin x$$

$$= \frac{1}{1+\sin^4 x} \cdot 2 \sin x \cdot \cos x$$

$$\Rightarrow \frac{\sin 2x}{1+\sin^4 x}$$

• $y = \sec(\tan^{-1} x)$, where $x=1$.

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx} (\sec(\tan^{-1} x))$$

$$= \sec(\tan^{-1} x) \cdot \tan(\tan^{-1} x) \cdot \frac{d}{dx} (\tan^{-1} x)$$

$$= \sec(\tan^{-1} x) \tan(\tan^{-1} x) \cdot \frac{1}{1+x^2}$$

For $x=1$,

$$\frac{dy}{dx} = \sec(\tan^{-1} 1) \cdot \tan(\tan^{-1} 1) \left(\frac{1}{1+1} \right)$$

$$\Rightarrow \sec\left(\frac{\pi}{4}\right) \cdot \tan\left(\frac{\pi}{4}\right) \cdot \frac{1}{2}$$

$$= \sqrt{2} \cdot 1 \cdot \frac{1}{2} = \frac{1}{\sqrt{2}}$$

Methods of Differentiation :-

① Differentiation by substitution ! -

Following substitutions are normally used to simplify these expressions -

$$\text{i) } \sqrt{\alpha^2 + a^2} \Rightarrow \alpha = a \tan \theta \text{ or } a \cot \theta$$

$$\text{ii) } \sqrt{a^2 - \alpha^2} \Rightarrow \alpha = a \sin \theta \text{ or } a \cos \theta$$

$$\text{iii) } \sqrt{\alpha^2 - a^2} \Rightarrow \alpha = a \sec \theta \text{ or } a \cosec \theta$$

$$\text{iv) } \sqrt{\frac{\alpha+a}{\alpha-a}} \Rightarrow \alpha = a \operatorname{cosec} \theta$$

e.g:- $y = \cos^{-1}(4x^3 - 3x)$

$$\text{Let } \alpha = \cos \theta \Rightarrow \theta = \cos^{-1} \alpha$$

$$\therefore y = \cos^{-1}(4 \cos^3 \theta - 3 \cos \theta)$$

$$= \cos^{-1}(\cos 3\theta)$$

$$= 3\theta = 3\cos^{-1} \alpha$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{d\alpha}(3\cos^{-1} \alpha) = \frac{-3}{\sqrt{1-\alpha^2}}$$

$$y = \tan^{-1}\left(\frac{\sqrt{\alpha} - \alpha}{1 + \alpha^{3/2}}\right)$$

$$\text{Put } \sqrt{\alpha} = \tan \alpha \text{ and } \alpha = \tan \beta$$

$$\Rightarrow \alpha = \tan^{-1} \sqrt{\alpha} \text{ and } \beta = \tan^{-1} \alpha$$

$$\therefore y = \tan^{-1}\left(\frac{\sqrt{\alpha} - \alpha}{1 + \alpha^{3/2}}\right) = \tan^{-1}\left(\frac{\sqrt{\alpha} - \alpha}{1 + \sqrt{\alpha} \cdot \alpha}\right)$$

$$= \tan^{-1}\left(\frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \cdot \tan \beta}\right)$$

$$\Rightarrow \tan^{-1}(\tan(\alpha - \beta))$$

$$\Rightarrow \alpha - \beta = \tan^{-1} \sqrt{\alpha} - \tan^{-1} \alpha$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{d\alpha} (\tan^{-1} \sqrt{\alpha} - \tan^{-1} \alpha)$$

$$\begin{aligned}
 &= \frac{1}{1+(\sqrt{\alpha})^2} \cdot \frac{d}{d\alpha} \sqrt{\alpha} = \frac{1}{1+\alpha} \\
 &= \frac{1}{1+\alpha} \cdot \frac{1}{2} \alpha^{-1/2} = \frac{1}{2\sqrt{\alpha}(1+\alpha)} \\
 &= \frac{1}{2\sqrt{\alpha}(1+\alpha)} - \frac{1}{1+\alpha^2}
 \end{aligned}$$

$$y = \tan^{-1} \left(\frac{\alpha - \sqrt{1-\alpha^2}}{\alpha + \sqrt{1-\alpha^2}} \right)$$

Put $\alpha = \sin \theta \Rightarrow d\alpha = \cos \theta d\theta$

$$\therefore y = \tan^{-1} \left(\frac{\sin \theta - \sqrt{1-\sin^2 \theta}}{\sin \theta + \sqrt{1-\sin^2 \theta}} \right)$$

$$\Rightarrow \tan^{-1} \left(\frac{\sin \theta - \cos \theta}{\sin \theta + \cos \theta} \right)$$

$$\Rightarrow \tan^{-1} \left(\frac{\tan \theta - 1}{\tan \theta + 1} \right)$$

$$\Rightarrow \tan^{-1} \left(\frac{\tan \theta - \tan \pi/4}{1 + \tan \theta \cdot \tan \pi/4} \right)$$

$$\Rightarrow \tan^{-1} \left(\tan \left(\theta - \frac{\pi}{4} \right) \right)$$

$$\Rightarrow \theta - \frac{\pi}{4} = \sin^{-1} \alpha - \frac{\pi}{4}$$

$$\Rightarrow \frac{dy}{d\alpha} = \frac{d}{d\alpha} \left(\sin^{-1} \alpha - \frac{\pi}{4} \right) = \frac{1}{\sqrt{1-\alpha^2}} - 0 = \frac{1}{\sqrt{1-\alpha^2}}$$

$$y = \sin^{-1} \left(\frac{2\alpha}{1+\alpha^2} \right) + \sec^{-1} \left(\frac{1+\alpha^2}{1-\alpha^2} \right)$$

$$= \sin^{-1} \left(\frac{2\alpha}{1+\alpha^2} \right) + \cos^{-1} \left(\frac{1-\alpha^2}{1+\alpha^2} \right)$$

Put $\alpha = \tan \theta \Rightarrow \theta = \tan^{-1} \alpha$

$$\therefore y = \sin^{-1} \left(\frac{2\tan \theta}{1+\tan^2 \theta} \right) + \cos^{-1} \left(\frac{1-\tan^2 \theta}{1+\tan^2 \theta} \right)$$

$$= \sin^{-1} (\sin 2\theta) + \cos^{-1} (\cos 2\theta)$$

$$= 2\theta + 2\theta = 4\theta = 4\tan^{-1} \alpha$$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(\ln \tan^{-1} x) = \frac{1}{1+x^2}$$

② Differentiation using Logarithms :-

when a function appears as an exponent of another function we make use of logarithms.

e.g. $y = (\sin x)^{\tan x}$

Taking logarithm on both sides, we have -

$$\ln y = \ln (\sin x)^{\tan x}$$

$$\Rightarrow \ln y = \tan x \cdot \ln(\sin x)$$

Diff. both sides w.r.t x , we obtain -

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}(\tan x \cdot \ln(\sin x))$$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = \ln(\sin x) \frac{d}{dx} \tan x + \tan x \frac{d}{dx}(\ln \sin x)$$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = \sec^2 x \cdot \ln(\sin x) + \tan x \cdot \frac{1}{\sin x} \cdot \cos x$$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = \sec^2 x \ln(\sin x) + \tan x \cdot \cot x$$

$$\Rightarrow \frac{dy}{dx} = y (\sec^2 x \ln(\sin x) + 1)$$

$$\Rightarrow \frac{dy}{dx} = (\sin x)^{\tan x} (1 + \sec^2 x \ln(\sin x))$$

$$y = \frac{(x-1)^2 \sqrt{3x-1}}{x^7 (6-7x^2)^{3/2}}$$

Taking logarithms on both sides, we have -

$$\ln y = \ln \frac{(x-1)^2 \sqrt{3x-1}}{x^7 (6-7x^2)^{3/2}}$$

$$\Rightarrow \ln y = \ln(x-1)^2 + \ln \sqrt{3x-1} - \ln x^7 - \ln(6-7x^2)^{3/2}$$

$$\Rightarrow 2\ln(x-1) + \frac{1}{2}\ln(3x-1) - 7\ln x - \frac{3}{2}\ln(6-7x^2)$$

Diff. both sides w.r.t α , we have -

$$\begin{aligned}\frac{1}{y} \frac{dy}{dx} &= 2\left(\frac{1}{\alpha-1}\right) \frac{d}{dx}(\alpha-1) + \frac{1}{2}\left(\frac{1}{3\alpha-1}\right) \frac{d}{dx}(3\alpha-1) - 7\frac{1}{\alpha} \\ &\quad - \frac{3}{2}\left(\frac{1}{6-7\alpha^2}\right) \frac{d}{dx}(6-7\alpha^2) \\ &= \frac{2}{\alpha-1} + \frac{3}{2(3\alpha-1)} - \frac{7}{\alpha} + \frac{21\alpha}{6-7\alpha^2}\end{aligned}$$

$$\Rightarrow \frac{dy}{dx} = y \left(\frac{2}{\alpha-1} + \frac{3}{2(3\alpha-1)} - \frac{7}{\alpha} + \frac{21\alpha}{6-7\alpha^2} \right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{(\alpha-1)^2 \sqrt{3\alpha-1}}{\alpha^7 (6-7\alpha^2)^{3/2}} \left[\frac{2}{\alpha-1} + \frac{3}{2(3\alpha-1)} - \frac{7}{\alpha} + \frac{21\alpha}{6-7\alpha^2} \right]$$

③ Differentiation of Implicit Functions :-

If $f(x, y) = 0$ is an implicit function then in order to find $\frac{dy}{dx}$, we differentiate each term w.r.t x regarding y as a function of x and then collect terms in $\frac{dy}{dx}$.

e.g:- $y^2 = \cos(\alpha+y)$, find $\frac{dy}{dx}$.

Diff. both sides w.r.t α , we get -

$$\Rightarrow 2y \frac{dy}{dx} = -\sin(\alpha+y) \frac{d}{dx}(\alpha+y)$$

$$\Rightarrow 2y \frac{dy}{dx} = -\sin(\alpha+y) \left(1 + \frac{dy}{dx} \right)$$

$$\Rightarrow 2y \frac{dy}{dx} + \sin(\alpha+y) \frac{dy}{dx} = -\sin(\alpha+y)$$

$$\Rightarrow [2y + \sin(\alpha+y)] \frac{dy}{dx} = -\sin(\alpha+y)$$

$$\Rightarrow \frac{dy}{dx} = \frac{-\sin(\alpha+y)}{2y + \sin(\alpha+y)}$$

- Find $\frac{dy}{dx}$ if $x^2 + y^2 - a^2 = 0$.

$$x^2 + y^2 - a^2 = 0$$

Diff. both sides w.r.t x , we have -

$$2x + 2y \frac{dy}{dx} = 0$$

$$\Rightarrow 2y \frac{dy}{dx} = -2x \Rightarrow \frac{dy}{dx} = -\frac{x}{y}$$

- $y^3 + 3x^2y - 2x = 10$, find $\frac{dy}{dx}$.

$$y^3 + 3x^2y - 2x = 10$$

Diff. both sides w.r.t x , we have -

$$\frac{d}{dx}(y^3 + 3x^2y - 2x) = \frac{d}{dx}(10)$$

$$\Rightarrow 3y^2 \frac{dy}{dx} + 3(2xy + x^2 \frac{dy}{dx}) - 2 = 0$$

$$\Rightarrow (3y^2 + 3x^2) \frac{dy}{dx} + 6xy - 2 = 0$$

$$\Rightarrow 3(y^2 + x^2) \frac{dy}{dx} = 2 - 6xy$$

$$\Rightarrow \frac{dy}{dx} = \frac{2(1 - 3xy)}{3(x^2 + y^2)}$$

- $\log(x+y) - 2xy = 0$, then $\frac{dy}{dx}$ at $x=0$.

$$\log(x+y) - 2xy = 0$$

Diff. both sides w.r.t x , we have -

$$\frac{d}{dx}(\log(x+y) - 2xy) = 0$$

$$\Rightarrow \left(\frac{1}{x+y}\right)\left(1 + \frac{dy}{dx}\right) - 2\left(y + x \frac{dy}{dx}\right) = 0$$

$$\Rightarrow \left(\frac{1}{x+y} - 2x\right)\left(\frac{dy}{dx}\right) + \frac{1}{x+y} - 2y = 0$$

$$\Rightarrow \left(\frac{1}{x+y} - 2x\right)\left(\frac{dy}{dx}\right) = 2y - \left(\frac{1}{x+y}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{2y - (\frac{1}{x+y})}{(\frac{1}{x+y}) - 2x} = \frac{2y(x+y) - 1}{1 - 2x(x+y)}$$

At $x=0$, $\frac{dy}{dx} = \frac{2y(0+y)-1}{1-0} = \frac{2y^2-1}{1}$

Now $\log(x+y) - 2xy = 0$

$$\Rightarrow \log(0+y) - 0 = 0$$

$$\Rightarrow \log y = 0 \Rightarrow y=1 \text{ i.e. } y(0)=1$$

$$\therefore \left[\frac{dy}{dx} \right]_{x=0} = \frac{2 \cdot 1 - 1}{1} = \frac{2-1}{1} = 1$$

$$\Rightarrow y'(0) = 1$$

(4) Differentiation of Parametric Functions:-

Suppose $x = \phi(t)$ and $y = \psi(t)$

where ϕ and ψ are two differentiable funⁿ of t . Then

$$\frac{dy}{dx} = \left(\frac{dy}{dt} \right) \cdot \left(\frac{dt}{dx} \right) = \left(\frac{dy}{dt} \right) / \left(\frac{dx}{dt} \right)$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{\psi'(t)}{\phi'(t)}}$$

e.g:- Find $\frac{dy}{dx}$ if $x = a(\cos t + t \sin t)$
and $y = a(\sin t - t \cos t)$.

$$\frac{dx}{dt} = \frac{d}{dt} a(\cos t + t \sin t)$$

$$= a(-\sin t + \sin t + t \cos t) = at \cos t$$

$$\frac{dy}{dt} = \frac{d}{dt} a(\sin t - t \cos t)$$

$$= a (\text{cosec} - \text{cosec} + t \sin t) = at \sin t$$

$$\therefore \frac{dy}{dx} = \left(\frac{dy}{dt} \right) / \left(\frac{dx}{dt} \right) = \frac{at \sin t}{at \cosec t} = \frac{\sin t}{\cosec t} = \tan t$$

• If $x = e^{\cos 2\theta}$ and $y = e^{\sin 2\theta}$, then find $\frac{dy}{dx}$.

$$\frac{dx}{d\theta} = \frac{d}{d\theta} (e^{\cos 2\theta}) = e^{\cos 2\theta} (-2 \sin 2\theta)$$

$$\frac{dy}{d\theta} = \frac{d}{d\theta} (e^{\sin 2\theta}) = e^{\sin 2\theta} (2 \cos 2\theta)$$

$$\therefore \frac{dy}{dx} = \left(\frac{dy}{d\theta} \right) / \left(\frac{dx}{d\theta} \right) = \frac{2 \cos 2\theta \cdot e^{\sin 2\theta}}{-2 \sin 2\theta \cdot e^{\cos 2\theta}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-\cos 2\theta \cdot e^{\sin 2\theta}}{\sin 2\theta \cdot e^{\cos 2\theta}}$$

$$\text{Now, } x = e^{\cos 2\theta} \Rightarrow \ln x = \cos 2\theta$$

$$y = e^{\sin 2\theta} \Rightarrow \ln y = \sin 2\theta$$

$$\therefore \frac{dy}{dx} = \frac{-y \ln x}{x \ln y}$$

⑤ Differentiation with respect to a function:-

Suppose $y = f(x)$ and $z = g(x)$ be two differentiable functions. To find the derivative of y w.r.t z we regard x as a parameter and find $f'(x) = \frac{dy}{dx}$ and $g'(x) = \frac{dz}{dx}$, then

$$\frac{dy}{dz} = \left(\frac{dy}{dx} \right) \left(\frac{dx}{dz} \right) = \left(\frac{dy}{dx} \right) / \left(\frac{dz}{dx} \right)$$

$$\Rightarrow \boxed{\frac{dy}{dz} = \frac{f'(x)}{g'(x)}}$$

e.g. Differentiate $\sec^{-1}\left(\frac{1}{2x^2-1}\right)$ w.r.t $\sqrt{1-x^2}$.

Let $y = \sec^{-1}\left(\frac{1}{2x^2-1}\right) = \cos^{-1}(2x^2-1)$

$$\Rightarrow y' = \frac{dy}{dx} = \frac{d}{dx} \cos^{-1}(2x^2-1)$$

$$= \frac{-1}{\sqrt{1-(2x^2-1)^2}} (4x)$$

$$= \frac{-4x}{\sqrt{1-4x^4-1+4x^2}}$$

$$= \frac{-4x}{\sqrt{4x^2-4x^4}}$$

$$= \frac{-4x}{2x\sqrt{1-x^2}} = \frac{-2}{\sqrt{1-x^2}}$$

Now, $z = \sqrt{1-x^2}$

$$\Rightarrow z' = \frac{dz}{dx} = \frac{d}{dx} \sqrt{1-x^2} = \frac{1}{2\sqrt{1-x^2}} \cdot (-2x) = \frac{-x}{\sqrt{1-x^2}}$$

$$\therefore \frac{dy}{dz} = \left(\frac{dy}{dx}\right) / \left(\frac{dz}{dx}\right) = \frac{-2}{\sqrt{1-x^2}} / \frac{-x}{\sqrt{1-x^2}} = 2/x$$

Differentiate $\tan^{-1}x$ w.r.t $\cos^{-1}x$.

Let $y = \tan^{-1}x$ and $z = \cos^{-1}x$

Now $\frac{dy}{dx} = \frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$

$$\frac{dz}{dx} = \frac{d}{dx}(\cos^{-1}x) = \frac{-1}{\sqrt{1-x^2}}$$

$$\therefore \frac{dy}{dz} = \left(\frac{dy}{dx}\right) / \left(\frac{dz}{dx}\right) = \left(\frac{1}{1+x^2}\right) / \left(\frac{-1}{\sqrt{1-x^2}}\right)$$

$$\Rightarrow \frac{dy}{dz} = -\frac{\sqrt{1-x^2}}{(1+x^2)}$$

Second Order Derivatives :-

If $y = f(\alpha)$ then $\frac{dy}{d\alpha} = f'(\alpha)$ is a function of α or constant which can be differentiated once again then $\frac{dy}{d\alpha} = f'(\alpha)$ is the first order derivatives of y w.r.t α .

Now, $\frac{d^2y}{d\alpha^2} = \frac{d}{d\alpha} \left(\frac{dy}{d\alpha} \right) = f''(\alpha)$ is the second order derivative of y w.r.t α . i.e the derivative of $\frac{dy}{d\alpha}$ w.r.t α .

Hence, $\frac{dy}{d\alpha} = y'(\alpha) = y_1 = f'(\alpha) = D(y)$

and $\frac{d^2y}{d\alpha^2} = y''(\alpha) = y_2 = f''(\alpha) = D^2(y)$

e.g.

Find y_2 if $y = \alpha^5 + 4\alpha^3 - 2\alpha^2 + 1$

$$\text{Now } y = \alpha^5 + 4\alpha^3 - 2\alpha^2 + 1$$

$$\Rightarrow y_1 = \frac{d}{d\alpha} (\alpha^5 + 4\alpha^3 - 2\alpha^2 + 1) = 5\alpha^4 + 12\alpha^2 - 4\alpha$$

$$\text{and } y_2 = \frac{d}{d\alpha} (5\alpha^4 + 12\alpha^2 - 4\alpha) = 20\alpha^3 + 24\alpha - 4$$

• If $\alpha = \sin t$ and $y = \sin(Pt)$, then show that

$$(1-\alpha^2) \frac{d^2y}{d\alpha^2} - \alpha \frac{dy}{d\alpha} + P^2 y = 0.$$

Hence $\alpha = \sin t$ and $y = \sin(Pt)$

$$\Rightarrow t = \sin^{-1}\alpha = \sin(P\sin^{-1}\alpha)$$

$$\therefore y = \sin(P\sin^{-1}\alpha)$$

$$\Rightarrow y_1 = \cos(P\sin^{-1}\alpha) \cdot \frac{P}{\sqrt{1-\alpha^2}}$$

$$\Rightarrow y_1^2 = \left(\frac{dy}{d\alpha} \right)^2 = P^2 \cos^2(P\sin^{-1}\alpha) \frac{1}{(1-\alpha^2)}$$

$$\begin{aligned} \Rightarrow (1-\alpha^2) \left(\frac{dy}{dx} \right)^2 &= P^2 \cos^2(P \sin^{-1} \alpha) \\ &\Rightarrow P^2 [1 - \sin^2(P \sin^{-1} \alpha)] \\ &\Rightarrow P^2 - P^2 \sin^2(P \sin^{-1} \alpha) \\ &\Rightarrow P^2 - P^2 y^2 \end{aligned}$$

Dif. w.r.t α , we have -

$$\begin{aligned} \Rightarrow (1-\alpha^2) 2 \left(\frac{dy}{dx} \right) \left(\frac{d^2y}{dx^2} \right) - 2\alpha \left(\frac{dy}{dx} \right)^2 &= -P^2 2y \frac{dy}{dx} \\ \Rightarrow 2 \left(\frac{dy}{dx} \right) \left[(1-\alpha^2) \left(\frac{d^2y}{dx^2} \right) - \alpha \left(\frac{dy}{dx} \right) \right] &= -P^2 2y \frac{dy}{dx} \\ \Rightarrow (1-\alpha^2) \left(\frac{d^2y}{dx^2} \right) - \alpha \frac{dy}{dx} + P^2 y &= 0. \end{aligned}$$

• $y = \alpha^2 \cos \alpha$, then Prove that

$$\alpha^2 y_2 - 4\alpha y_1 + (\alpha^2 + 6)y = 0$$

$$y = \alpha^2 \cos \alpha$$

$$\begin{aligned} \Rightarrow y_1 &= \cos \alpha \cdot 2\alpha + \alpha^2 (-\sin \alpha) \\ &= 2\alpha \cos \alpha - \alpha^2 \sin \alpha \end{aligned}$$

$$\begin{aligned} \Rightarrow y_2 &= \cos \alpha \cdot 2 + 2\alpha (-\sin \alpha) - [\sin \alpha \cdot 2\alpha + \alpha^2 \cos \alpha] \\ &= 2\cos \alpha - 2\alpha \sin \alpha - 2\alpha \sin \alpha - \alpha^2 \cos \alpha \\ &= (2-\alpha^2) \cos \alpha - 4\alpha \sin \alpha \end{aligned}$$

$$\therefore \alpha^2 y_2 - 4\alpha y_1 + (\alpha^2 + 6)y$$

$$\begin{aligned} &\Rightarrow \alpha^2 [(2-\alpha^2) \cos \alpha - 4\alpha \sin \alpha] - 4\alpha [2\alpha \cos \alpha - \alpha^2 \sin \alpha] \\ &\quad + (\alpha^2 + 6)\alpha^2 \cos \alpha. \end{aligned}$$

$$\begin{aligned} &= 2\alpha^2 \cos \alpha - \alpha^4 \cos \alpha - 4\alpha^3 \sin \alpha - 8\alpha^2 \cos \alpha + 4\alpha^3 \sin \alpha \\ &\quad + \alpha^4 \cos \alpha + 6\alpha^2 \cos \alpha \end{aligned}$$

$$\Rightarrow (2\alpha^2 - 8\alpha^2 + 6\alpha^2) \cos \alpha - \frac{9}{4} \cdot (n+1)(n+2) \cos \alpha = 0$$

$$= 0$$

Partial Differentiation :-

Functions of two or more variables:-

when there is more than one dependent variables in a function, we consider

$$u = f(x, y)$$

where u be a variable & depends on both x and y .

Partial Differentiation!

Let $u = f(x, y)$, then the partial differentiation of u w.r.t x is defined as differentiation of u w.r.t x as treating 'y' as constant and is denoted by $\frac{\partial u}{\partial x}$.

$$\therefore \frac{\partial u}{\partial x} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

$\Rightarrow \frac{\partial u}{\partial x}$: Diff. of u w.r.t x treating y as a constant.

similarly, Partial differentiation of u w.r.t y is defined as differentiation of u w.r.t y treating x as constant and is denoted by $\frac{\partial u}{\partial y}$.

$$\therefore \frac{\partial u}{\partial y} = \lim_{\Delta y \rightarrow 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

$\Rightarrow \frac{\partial u}{\partial y}$ = Diff. of u w.r.t y treating x as a constant.

$$\text{Second order Partial Derivatives:}$$

$$\Rightarrow \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \right) = \frac{\partial^2 u}{\partial x^2} \quad \Rightarrow \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial x} \right) = \frac{\partial^2 u}{\partial y \partial x}$$

$$\Rightarrow \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial y} \right) = \frac{\partial^2 u}{\partial x \partial y} \quad \Rightarrow \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial y} \right) = \frac{\partial^2 u}{\partial y^2}$$

And for all the functions,

$$\boxed{\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}}$$

E.g. • $u = x + y$ • $u = x^3 + y^3$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x} (x+y) = 1+0=1 \quad \frac{\partial u}{\partial x} = \frac{\partial}{\partial x} (x^3+y^3)$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y} (x+y) = 0+1=1 \quad = 3x^2$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y} (x^3+y^3) = 3y^2$$

• $u = e^{2x+3y}$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x} (e^{2x+3y}) = e^{2x+3y} \quad \frac{\partial}{\partial x} (2x+3y) = 2e^{2x+3y}$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y} (e^{2x+3y}) = e^{2x+3y} \quad \frac{\partial}{\partial y} (2x+3y) = 3e^{2x+3y}$$

• $u = x^3 + 4x^2y + 5xy^2 + y^3$

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial}{\partial x} (x^3 + 4x^2y + 5xy^2 + y^3) \\ &= 3x^2 + 8xy + 5y^2 \end{aligned}$$

$$\begin{aligned} \frac{\partial u}{\partial y} &= \frac{\partial}{\partial y} (x^3 + 4x^2y + 5xy^2 + y^3) \\ &= 4x^2 + 10xy + 3y^2 \end{aligned}$$

$u = \alpha^3 \tan y$, find $\frac{\partial^2 u}{\partial x^2}$, $\frac{\partial^2 u}{\partial y^2}$ and $\frac{\partial^2 u}{\partial x \partial y}$, $\frac{\partial^2 u}{\partial y \partial x}$.

$$u = \alpha^3 \tan y$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial}{\partial x} (\alpha^3 \tan y) = 3\alpha^2 \tan y$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} (3\alpha^2 \tan y) = 6\alpha^2 \sec^2 y$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y} (\alpha^3 \tan y) = \alpha^3 \sec^2 y$$

$$\Rightarrow \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y} (\alpha^3 \sec^2 y) = \alpha^3 2 \sec y (\sec y \cdot \tan y)$$
$$= 2\alpha^3 \sec^2 y \cdot \tan y$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x} (\alpha^3 \sec^2 y) = 3\alpha^2 \sec^2 y$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y} (3\alpha^2 \sec^2 y) = 3\alpha^2 \sec^2 y$$

$$\therefore \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$