

**LEARNING MATERIAL
ON
DETERMINANT & MATRICES**

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DETERMINANT AND MATRICES

DETERMINANT

The credit for the discovery of the subject of determinant goes to the German mathematician, Gauss. After the introduction of determinants, solving a system of simultaneous linear equations becomes much simpler.

Definition:

Determinant is a square arrangement of numbers (real or complex) within two vertical lines.

Example :

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \text{ is a determinant}$$

Determinant of second order:

The symbol $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ consisting of 4 numbers a, b, c and arranged in two rows and two columns is called a determinant of second order.

The numbers a, b, c, and d are called elements of the determinant. The value of the determinant is $\Delta = ad - bc$

Examples:

$$1. \begin{vmatrix} 2 & 3 \\ 5 & 1 \end{vmatrix} = (2)(1) - (5)(3) = 2 - 15 = -13$$

$$2. \begin{vmatrix} 4 & 6 \\ 3 & -5 \end{vmatrix} = (4)(-5) - (6)(3) = -20 - 18 = -38$$

Determinant of third order:

The expression $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ consisting of

nine elements arranged in three rows and three columns is called a determinant of third order

The value of the determinant is obtained by expanding the determinant along the first row

$$\begin{aligned} \Delta &= a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \\ &= a_1 (b_2 c_3 - b_3 c_2) - b_1 (a_2 c_3 - a_3 c_2) + c_1 (a_2 b_3 - a_3 b_2) \end{aligned}$$

Note: The determinant can be expanded along any row or column.

Examples:

$$\begin{aligned} (1) \quad \begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 5 & 2 & 1 \end{vmatrix} &= 1(1-8) - 2(2-20) + 3(4-5) \\ &= 1(-7) - 2(-18) + 3(-1) \\ &= -7 + 36 - 3 \\ &= -10 + 36 = 26 \end{aligned}$$

$$\begin{aligned} (2) \quad \begin{vmatrix} 3 & 0 & 1 \\ 2 & -3 & 4 \\ 1 & -1 & -2 \end{vmatrix} &= 3(6+4) + 1(-2+3) \\ &= 3(10) + 1(1) \\ &= 30 + 1 \\ &= 31 \end{aligned}$$

Minor of an element

Definition :

Minor of an element is a determinant obtained by deleting the row and column in which that element occurs. The Minor of i^{th} row j^{th} Column element is denoted by m_{ij}

Example:

$$\begin{vmatrix} 1 & -1 & 3 \\ 0 & 4 & 2 \\ 11 & 5 & -3 \end{vmatrix}$$

$$\text{Minor of } 3 = \begin{vmatrix} 0 & 4 \\ 11 & 5 \end{vmatrix} = 0-44 = -44$$

$$\text{Minor of } 0 = \begin{vmatrix} -1 & 3 \\ 5 & -3 \end{vmatrix} = 3-15 = -12$$

Cofactor of an element**Definition :**

Co-factor of an element in i^{th} row, j^{th} column is the signed minor of i^{th} row J^{th} Column element and is denoted by A_{ij} .

$$(i.e) A_{ij} = (-1)^{i+j} m_{ij}$$

The sign is attached by the rule $(-1)^{i+j}$

Example

$$\begin{vmatrix} 3 & -2 & 4 \\ 2 & 1 & 0 \\ 7 & 11 & 6 \end{vmatrix}$$

$$\text{Co-factor of } -2 = (-1)^{1+2} \begin{vmatrix} 2 & 0 \\ 7 & 6 \end{vmatrix} = (-1)^3 (12) = -12$$

$$\text{Co-factor of } 7 = (-1)^{3+1} \begin{vmatrix} -2 & 4 \\ 1 & 0 \end{vmatrix} = (-1)^4 (0-4) = -4$$

Properties of Determinants:**Property 1:**

The value of a determinant is unaltered when the rows and columns are interchanged.

$$(i.e) \text{ If } \Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \text{ and } \Delta^T = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

then $\Delta^T = \Delta$

Property 2:

If any two rows or columns of a determinant are interchanged the value of the determinant is changed in its sign.

$$\text{If } \Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \text{ and } \Delta_1 = \begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix},$$

then $\Delta_1 = -\Delta$

Note: R_1 and R_2 are interchanged.

Property 3:

If any two rows or columns of a determinant are identical, then the value of the determinant is zero.

$$\text{(i.e) The value of } \begin{vmatrix} a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \text{ is zero} \quad \text{Since } R_1 \equiv R_2$$

Property 4:

If each element of a row or column of a determinant is multiplied by any number $K \neq 0$, then the value of the determinant is multiplied by the same number K .

$$\text{If } \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\text{and } \Delta_1 = \begin{vmatrix} Ka_1 & Kb_1 & Kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

then $\Delta_1 = K\Delta$

Property 5:

If each element of a row or column is expressed as the sum of two elements, then the determinant can be expressed as the sum of two determinants of the same order.

$$(i.e) \text{ If } \Delta = \begin{vmatrix} a_1 + d_1 & b_1 + d_2 & c_1 + d_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

$$\text{then } \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} d_1 & d_2 & d_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Property 6:

If each element of a row or column of a determinant is multiplied by a constant $K \neq 0$ and then added to or subtracted from the corresponding elements of any other row or column then the value of the determinant is unaltered.

$$\text{Let } \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\Delta_1 = \begin{vmatrix} a_1 + ma_2 + na_3 & b_1 + mb_2 + nb_3 & c_1 + mc_2 + nc_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} ma_2 & mb_2 & mc_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} na_3 & nb_3 & nc_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= \Delta + m \begin{vmatrix} a_2 & b_2 & c_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + n \begin{vmatrix} a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\Delta_1 = \Delta + m (0) + n (0) = \Delta$$

Property 7:

In a given determinant if two rows or columns are identical for $x = a$, then $(x-a)$ is a factor of the determinant.

$$\text{Let } \Delta = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}$$

$$\text{For } a=b, \Delta = \begin{vmatrix} 1 & 1 & 1 \\ b & b & c \\ b^3 & b^3 & c^3 \end{vmatrix} = 0 \quad [C_1, \text{ and } C_2 \text{ are identical}]$$

$\therefore (a-b)$ is a factor of Δ

Notation :

Usually the three rows of the determinant first row, second row and third row are denoted by R_1 , R_2 and R_3 respectively and the columns by C_1 , C_2 and C_3

If we have to interchange two rows say R_1 and R_2 the symbol double sided arrow will be used. We will write like this $R_1 \leftrightarrow R_2$ it should be read as “is interchanged with” similarly for columns $C_2 \leftrightarrow C_3$.

If the elements of R_2 are subtracted from the corresponding elements of R_1 , then we write $R_1 - R_2$ similarly for columns also.

If the elements of one column say C_1 , ‘m’ times the element of C_2 and n times that of C_3 are added, we write like this $C_1 \rightarrow C_1 + m C_2 + n C_3$. Here one sided arrow is to be read as “is changed to”

Solution of simultaneous equations using Cramer’s rule:

Consider the linear equations.

$$a_1x + b_1y = c_1$$

$$a_2x + b_2y = c_2$$

$$\text{let } \Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

$$\Delta_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$$

$$\Delta_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

Then $x = \frac{\Delta_x}{\Delta}$ and $y = \frac{\Delta_y}{\Delta}$, provided $\Delta \neq 0$

x, y are unique solutions of the given equations. This method of solving the line equations is called Cramer's rule.

Similarly for a set of three simultaneous equations in x, y and z

$$a_1 x + b_1 y + c_1 z = d_1$$

$$a_2 x + b_2 y + c_2 z = d_2 \text{ and}$$

$$a_3 x + b_3 y + c_3 z = d_3, \text{ the solution of the system of equations,}$$

$$\text{by cramer's rule is given by, } x = \frac{\Delta_x}{\Delta}, y = \frac{\Delta_y}{\Delta} \text{ and } z = \frac{\Delta_z}{\Delta},$$

provided $\Delta \neq 0$

where

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \Delta_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

$$\Delta_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \text{ and } \Delta_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

WORKED EXAMPLESPART – A

1. Solve $\begin{vmatrix} x & 2 \\ x & 3x \end{vmatrix} = 0$

Solution:

$$\begin{vmatrix} x & 2 \\ x & 3x \end{vmatrix} = 0$$

$$3x^2 - 2x = 0$$

$$x(3x - 2) = 0$$

$$x = 0 \text{ or } x = \frac{2}{3}$$

2. Solve $\begin{vmatrix} x & 8 \\ 2 & x \end{vmatrix} = 0$

Solution:

$$\begin{vmatrix} x & 8 \\ 2 & x \end{vmatrix} = 0$$

$$x^2 - 16 = 0$$

$$x^2 = 16$$

$$x = \pm 4$$

3. Find the value of 'm' when $\begin{vmatrix} m & 2 & 1 \\ 3 & 4 & 2 \\ -7 & 3 & 0 \end{vmatrix} = 0$

Solution:

Given $\begin{vmatrix} m & 2 & 1 \\ 3 & 4 & 2 \\ -7 & 3 & 0 \end{vmatrix} = 0$

Expanding the determinant along, R_1 we have

$$m(0-6) - 2(0+14) + 1(9+28) = 0$$

$$m(-6) - 2(14) + 1(37) = 0$$

$$-6m - 28 + 37 = 0$$

$$-6m + 9 = 0$$

$$-6m = -9$$

$$m = \frac{9}{6} = \frac{3}{2}$$

4. Find the Co-factor of element 3 in the determinant $\begin{vmatrix} 1 & 2 & 0 \\ -1 & 3 & 4 \\ 5 & 6 & 7 \end{vmatrix}$

Solution:

$$\begin{aligned} \text{Cofactor of 3} &= A_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 0 \\ 5 & 7 \end{vmatrix} \\ &= (-1)^4 (7-0) = 7 \end{aligned}$$

PART – B

1. Using cramer's rule, solve the following simultaneous equations

$$x + y + z = 2$$

$$2x - y - 2z = -1$$

$$x - 2y - z = 1$$

Solution:

$$\begin{aligned} \Delta &= \begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & -2 \\ 1 & -2 & -1 \end{vmatrix} \\ &= 1(1-4) - 1(-2+2) + 1(-4+1) \\ &= 1(-3) - 1(0) + 1(-3) \\ &= -3 - 3 = -6 \neq 0 \end{aligned}$$

$$\Delta_x = \begin{vmatrix} 2 & 1 & 1 \\ -1 & -1 & -2 \\ 1 & -2 & -1 \end{vmatrix}$$

$$= 2 (1-4) -1 (1+2) +1 (2+1)$$

$$= 2 (-3) -1 (3) +1 (3)$$

$$= -6 -3 +3 = -6$$

$$\Delta_y = \begin{vmatrix} 1 & 2 & 1 \\ 2 & -1 & -2 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= 1 (1+2) -2 (-2+2) +1 (2+1)$$

$$= 1 (3) -2 (0) +1 (3)$$

$$= 3+3 = 6$$

$$\Delta_z = \begin{vmatrix} 1 & 1 & 2 \\ 2 & -1 & -1 \\ 1 & -2 & 1 \end{vmatrix}$$

$$= 1(-1-2) -1 (2+1) +2 (-4+1)$$

$$= -3-3-6 = -12$$

∴ By Cramer's rule,

$$x = \frac{\Delta_x}{\Delta} = \frac{-6}{-6} = 1 \quad y = \frac{\Delta_y}{\Delta} = \frac{6}{-6} = -1$$

$$z = \frac{\Delta_z}{\Delta} = \frac{-12}{-6} = 2$$

2. Using Cramer's rule solve: $-2y+3z-2x+1=0$

$$-x+y-z+5=0 \quad -2z \quad -4x+y = 4$$

Solution:

Rearrange the given equations in order

$$-2x-2y+3z = -1; -x+y-z = -5; -4x+y-2z = 4$$

$$\Delta = \begin{vmatrix} -2 & -2 & 3 \\ -1 & 1 & -1 \\ -4 & 1 & -2 \end{vmatrix}$$

$$= -2(-2+1) + 2(2-4) + 3(-1+4)$$

$$= -2(-1) + 2(-2) + 3(3)$$

$$= 2-4+9=7$$

$$\Delta_x = \begin{vmatrix} -1 & -2 & 3 \\ -5 & 1 & -1 \\ 4 & 1 & -2 \end{vmatrix}$$

$$= -1(-2+1) + 2(10+4) + 3(-5-4)$$

$$= 1+28-27$$

$$= 2$$

$$\Delta_y = \begin{vmatrix} -2 & -1 & 3 \\ -1 & -5 & -1 \\ -4 & 4 & -2 \end{vmatrix}$$

$$= -2(10+4) + 1(2-4) + 3(-4-20)$$

$$= -2(14) + 1(-2) + 3(-24)$$

$$= -28-2-72$$

$$= -102$$

$$\Delta_z = \begin{vmatrix} -2 & -2 & -1 \\ -1 & 1 & -5 \\ -4 & 1 & 4 \end{vmatrix}$$

$$= -2(4+5) + 2(-4-20) - 1(-1+4)$$

$$= -18-48-3$$

$$= -69$$

$$x = \frac{\Delta_x}{\Delta} = \frac{2}{7}, y = \frac{\Delta_y}{\Delta} = \frac{-102}{7}, \text{ and } z = \frac{\Delta_z}{\Delta} = \frac{-69}{7}$$

3. Using Cramer's rule solve

$$2x-3y=5$$

$$x-8=4y$$

Solution:

$$2x-3y = 5$$

$$x-4y = 8$$

$$\Delta = \begin{vmatrix} 2 & -3 \\ 1 & -4 \end{vmatrix} = (2)(-4) - (-3)(1) \\ = -8 + 3 = -5$$

$$\Delta_x = \begin{vmatrix} 5 & -3 \\ 8 & -4 \end{vmatrix} = (5)(-4) - (-3)(8) \\ = -20 + 24 = 4$$

$$\Delta_y = \begin{vmatrix} 2 & 5 \\ 1 & 8 \end{vmatrix} = 16 - 5 = 11$$

By Cramer's rule

$$x = \frac{\Delta_x}{\Delta} = \frac{4}{-5} = -\frac{4}{5}$$

$$y = \frac{\Delta_y}{\Delta} = \frac{11}{-5} = -\frac{11}{5}$$

PROBLEMS INVOLVING PROPERTIES OF DETERMINANTS

PART-A

1) Evaluate

$$\begin{vmatrix} 20 & 11 & 31 \\ 11 & -7 & 4 \\ 19 & 11 & 30 \end{vmatrix}$$

Solution:

$$\Delta = \begin{vmatrix} 20 & 11 & 31 \\ 11 & -7 & 4 \\ 19 & 11 & 30 \end{vmatrix} \\ = \begin{vmatrix} 31 & 11 & 31 \\ 4 & -7 & 4 \\ 30 & 11 & 30 \end{vmatrix} \quad C_1 \rightarrow C_1 + C_2$$

$$\Delta = 0 \quad \text{since } C_1 \equiv C_3$$

2) Without expanding, find the value of

$$\begin{vmatrix} 1 & -2 & 3 \\ 1 & -1 & 2 \\ 3 & -6 & 9 \end{vmatrix}$$

Solution:

$$\text{Let } \Delta = \begin{vmatrix} 1 & -2 & 3 \\ 1 & -1 & 2 \\ 3 & -6 & 9 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 3 \\ 1 & -1 & 2 \\ 3(1) & 3(-2) & 3(3) \end{vmatrix}$$

$$= 3 \begin{vmatrix} 1 & -2 & 3 \\ 1 & -1 & 2 \\ 1 & -2 & 3 \end{vmatrix}$$

$$= 3 (0) = 0, \text{ since } R_1 \equiv R_3$$

3) Evaluate

$$\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$$

Solution:

$$\text{Let } \Delta = \begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a+b+c & b+c \\ 1 & a+b+c & c+a \\ 1 & a+b+c & a+b \end{vmatrix} \quad C_2 \rightarrow C_2 + C_3$$

$$= (a+b+c) \begin{vmatrix} 1 & 1 & b+c \\ 1 & 1 & c+a \\ 1 & 1 & a+b \end{vmatrix}$$

$$= (a+b+c) (0) = 0, \text{ since } C_1 \equiv C_2$$

4) Prove that
$$\begin{vmatrix} x-y & y-z & z-x \\ y-z & z-x & x-y \\ z-x & x-y & y-z \end{vmatrix} = 0$$

Solution:

$$\text{L.H.S} = \begin{vmatrix} x-y & y-z & z-x \\ y-z & z-x & x-y \\ z-x & x-y & y-z \end{vmatrix}$$

$$\begin{vmatrix} x-y+y-z+z-x & y-z & z-x \\ y-z+z-x+x-y & z-x & x-y \\ z-x+x-y+y-z & x-y & y-z \end{vmatrix} \quad C_1 \rightarrow C_1 + C_2 + C_3$$

$$= \begin{vmatrix} 0 & y-z & z-x \\ 0 & z-x & x-y \\ 0 & x-y & y-z \end{vmatrix} = 0 = \text{R.H.S}$$

PART – B

1) Prove that
$$\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = (x-y)(y-z)(z-x)$$

Solution:

$$\text{L.H.S} = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

$$\begin{vmatrix} 0 & x-y & x^2-y^2 \\ 0 & y-z & y^2-z^2 \\ 1 & z & z^2 \end{vmatrix} \quad R_1 \rightarrow R_1 - R_2, R_2 \rightarrow R_2 - R_3$$

$$= (x-y) (y-z) \begin{vmatrix} 0 & 1 & x+y \\ 0 & 1 & y+z \\ 1 & z & z^2 \end{vmatrix}$$

$$= (x-y) (y-z) \begin{vmatrix} 1 & x+y \\ 1 & y+z \end{vmatrix} \text{ (expanded along the first column)}$$

$$= (x-y) (y-z) [1(y+z) - 1(x+y)]$$

$$= (x-y) (y-z) (z-x)$$

$$\text{L.H.S} = \text{R.H.S}$$

$$2) \text{ Prove that } \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a+b+c) (a-b) (b-c) (c-a)$$

Solution:

$$\text{L.H.S} = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}$$

$$\Delta = \begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ a^3-b^3 & b^3-c^3 & c^3 \end{vmatrix} \quad C_1 \rightarrow C_1 - C_2, C_2 \rightarrow C_2 - C_3$$

$$\Delta = \begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ (a-b)(a^2+ab+b^2) & (b-c)(b^2+bc+c^2) & c^3 \end{vmatrix}$$

$$\Delta = (a-b) (b-c) \begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & c \\ a^2+ab+b^2 & b^2+bc+c^2 & c^3 \end{vmatrix}$$

$$\Delta = (a-b) (b-c) \begin{vmatrix} 1 & 1 \\ a^2+ab+b^2 & b^2+bc+c^2 \end{vmatrix}$$

(expanded along the first row)

$$\begin{aligned}
&= (a-b)(b-c) \left[b^2 + bc + c^2 - (a^2 + ab + b^2) \right] \\
&= (a-b)(b-c) [b^2 + bc + c^2 - a^2 - ab - b^2] \\
&= (a-b)(b-c) [bc + c^2 - a^2 - ab] \\
&= (a-b)(b-c) [c^2 - a^2 + b(c-a)] \\
&= (a-b)(b-c) [(c+a)(c-a) + b(c-a)] \\
&= (a-b)(b-c)(c-a)[c+a+b] = \text{R.H.S}
\end{aligned}$$

3) Prove that
$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+a & 1 \\ 1 & 1 & 1+a \end{vmatrix} = a^2(3+a)$$

Solution:

$$\begin{aligned}
\text{L.H.S} &= \begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+a & 1 \\ 1 & 1 & 1+a \end{vmatrix} \\
&= \begin{vmatrix} 3+a & 3+a & 3+a \\ 1 & 1+a & 1 \\ 1 & 1 & 1+a \end{vmatrix} \quad R_1 \rightarrow R_1 + R_2 + R_3 \\
&= 3+a \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+a & 1 \\ 1 & 1 & 1+a \end{vmatrix} \\
&= (3+a) \begin{vmatrix} 0 & 0 & 1 \\ -a & a & 1 \\ 0 & -a & 1+a \end{vmatrix} \quad C_1 \rightarrow C_1 - C_2, C_2 \rightarrow C_2 - C_3 \\
&= (3+a) \begin{vmatrix} -a & a \\ 0 & -a \end{vmatrix} = (3+a)(a^2-0) \\
&= a^2(3+a) = \text{R.H.S}
\end{aligned}$$

4) Prove that
$$\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$$

Solution:

$$\begin{aligned} \text{L.H.S} &= \begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} \\ &= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} \quad R_1 \rightarrow R_1 + R_2 + R_3 \\ &= (a+b+c) \begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} \\ &= (a+b+c) \begin{vmatrix} 0 & 0 & 1 \\ a+b+c & -(a+b+c) & 2b \\ 0 & a+b+c & c-a-b \end{vmatrix} \begin{matrix} C_1 \rightarrow C_1 - C_2, \\ C_2 \rightarrow C_2 - C_3 \end{matrix} \\ &= (a+b+c) \begin{vmatrix} a+b+c & -(a+b+c) \\ 0 & (a+b+c) \end{vmatrix} \\ &= (a+b+c) [(a+b+c)^2 - 0] \\ &= (a+b+c)^3 = \text{R.H.S} \end{aligned}$$

5) Prove that
$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc \left[1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right]$$

Solution:

$$\text{Let } \Delta = \begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = \begin{vmatrix} a(\frac{1}{a}+1) & a(\frac{1}{a}) & a(\frac{1}{a}) \\ b(\frac{1}{b}) & b(\frac{1}{b}+1) & b(\frac{1}{b}) \\ c(\frac{1}{c}) & c(\frac{1}{c}) & c(\frac{1}{c}+1) \end{vmatrix}$$

$$= abc \begin{vmatrix} \frac{1}{a} + 1 & \frac{1}{a} & \frac{1}{a} \\ \frac{1}{b} & \frac{1}{b} + 1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c} + 1 \end{vmatrix}$$

$$= abc \begin{vmatrix} 1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} & 1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} & 1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \\ \frac{1}{b} & \frac{1}{b} + 1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c} + 1 \end{vmatrix} \quad R_1 \rightarrow R_1 + R_2 + R_3$$

$$= abc \left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \begin{vmatrix} \frac{1}{b} & \frac{1}{b} + 1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c} + 1 \end{vmatrix}$$

$$= abc \left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \begin{vmatrix} 0 & 0 & 1 \\ -1 & 1 & \frac{1}{b} \\ 0 & -1 & \frac{1}{c} + 1 \end{vmatrix} \quad \begin{matrix} C_1 \rightarrow C_1 - C_2, C_2 \rightarrow C_2 - C_3 \end{matrix}$$

$$= abc \left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \begin{vmatrix} -1 & 1 \\ 0 & -1 \end{vmatrix}$$

$$= abc \left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) [1-0]$$

$$\Delta = abc \left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$$

MATRICES

Introduction:

The term matrix was first introduced by a French mathematician Cayley in the year 1857. The theory of matrices is one of the powerful tools of mathematics not only in the field of higher mathematics but also in other branches such as applied sciences, nuclear physics, probability and statistics, economics and electrical circuits.

Definition:

A Matrix is a rectangular array of numbers arranged in to rows and columns enclosed by parenthesis or square brackets.

Example:

$$1. A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad 2. B = \begin{pmatrix} 2 & 1 & 0 \\ -5 & 6 & 7 \\ 1 & 0 & 8 \end{pmatrix}$$

Usually the matrices are denoted by capital letters of English alphabets A,B,C...,etc and the elements of the matrices are represented by small letters a,b,c,.etc.

Order of a matrix

If there are m rows and n columns in a matrix, then the order of the matrix is mxn or m by n.

$$\text{Example: } A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$$

A has two rows and three columns. We say that A is a matrix of order 2x3

Types of matrices

Row matrix:

A matrix having only one row and any number of columns is called a row matrix.

$$\text{Eg: } A = (1 \ 2 \ -3)$$

Column matrix

Matrix having only one column and any number of rows is called a column matrix.

$$\text{Eg: } B = \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix}$$

Square matrix

A matrix which has equal number of rows and columns is called a square matrix.

$$\text{Eg: } A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 4 \\ 3 & 2 & 6 \end{bmatrix} ; \quad B = \begin{bmatrix} 3 & -9 \\ 4 & 1 \end{bmatrix}$$

A is a square matrix of order 3

B is a square matrix of order 2

Null matrix (or) zero matrix or, void matrix:

If all the elements of a matrix are zero, then the matrix is called a null or zero matrix or void matrix it is denoted by 0.

$$\text{Eg: } 0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (2) 0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Diagonal matrix:

A square matrix with all the elements equal to zero except those in the leading diagonal is called a diagonal matrix

$$\text{Eg: } \begin{bmatrix} 5 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Unit matrix:

Unit matrix is a square matrix in which the principal diagonal elements are all ones and all the other elements are zeros.

It is denoted by I.

$$\text{Eg: } I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Here I_3 is a unit matrix of order 3.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

I_2 is a unit matrix of order 2.

Algebra of matrices:

Equality of two matrices:

Two matrices A and B are said to be equal if and only if order of A and order of B are equal and the corresponding elements of A and B are equal.

$$\text{Eg: if } A = \begin{bmatrix} 1 & 0 & 5 \\ -3 & 1 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} a & b & c \\ x & y & z \end{bmatrix}$$

then $A = B$ means $a=1 \quad b=0 \quad c=5$
 $x=-3 \quad y=1 \quad z=4$

Addition of matrices:

If A and B are any two matrices of the same order, then their sum $A+B$ is of the same order and is obtained by adding the corresponding elements of A and B.

$$\text{Eg: If } A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix} \quad \text{then}$$

$$A+B = \begin{bmatrix} 1+4 & 2+6 \\ 3+7 & 0+9 \end{bmatrix} = \begin{bmatrix} 5 & 8 \\ 10 & 9 \end{bmatrix}$$

Note : If the matrices are different order, addition is not possible.

Subtraction of matrices:

If A and B are any two matrices of the same order, then their difference A-B is of the same order and is obtained by subtracting the elements of B from the corresponding elements of A.

$$\begin{aligned} \text{Eg: If } A &= \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 4 & 1 \\ 2 & -1 \end{bmatrix} \quad \text{then} \\ A-B &= \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix} - \begin{bmatrix} 4 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1-4 & 3-1 \\ 2-2 & 0+1 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

Scalar multiplication of a matrix

If A is a given matrix, K is a number real or complex and $K \neq 0$ then KA is obtained by multiplying each element of A by K. It is called scalar multiplication of the matrix.

$$\begin{aligned} \text{Eg: if } A &= \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 5 \\ 3 & 1 & 6 \end{bmatrix} \quad \text{and } K=3 \\ KA &= 3 \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 5 \\ 3 & 1 & 6 \end{bmatrix} = \begin{bmatrix} 3 & 6 & -3 \\ 6 & 0 & 15 \\ 9 & 3 & 18 \end{bmatrix} \end{aligned}$$

Multiplication of two matrices:

Two matrices A and B are conformable for multiplication if and only if the number of columns in A is equal to the number of rows in B.

Note: If A is $m \times n$ matrix and B is $n \times p$ matrix then AB exists and is of order $m \times p$.

Method of multiplication:

$$\text{Let } A = \begin{bmatrix} a & b & c \\ 1 & 1 & 1 \\ a_2 & b_2 & c_2 \end{bmatrix}_{2 \times 3} \quad \text{and } B = \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix}_{3 \times 3}$$

Here the number of columns of matrix A is equal to the number of rows of matrix B. Hence AB can be found and the order is 2×3 .

Each element of the first row of AB is got by adding the product of the elements of first row of A with the corresponding elements of first, second and third columns of B. On similar lines, we can also get the second row of AB.

$$\begin{aligned}
 \text{(ie) } AB &= \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix} \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix} \\
 &= \begin{pmatrix} a_1x_1 + b_1x_2 + c_1x_3 & a_1y_1 + b_1y_2 + c_1y_3 & a_1z_1 + b_1z_2 + c_1z_3 \\ a_2x_1 + b_2x_2 + c_2x_3 & a_2y_1 + b_2y_2 + c_2y_3 & a_2z_1 + b_2z_2 + c_2z_3 \end{pmatrix}
 \end{aligned}$$

Note: (1) If A is of order 3×3 and B is of order 3×2 then AB is of order 3×2 but BA does not exist.

(2) If AB and BA are of same order they need not be equal. In general $AB \neq BA$.

Example:

$$\begin{aligned}
 \text{If } A &= \begin{bmatrix} 1 & 2 & 3 \\ 3 & 0 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} -1 & 2 \\ 3 & 4 \\ 4 & -3 \end{bmatrix} \text{ then} \\
 AB &= \begin{bmatrix} -1+6+12 & 2+8-9 \\ -3+0+8 & 6+0-6 \end{bmatrix} = \begin{bmatrix} 17 & 1 \\ 5 & 0 \end{bmatrix}
 \end{aligned}$$

Transpose of a matrix

If the rows and columns of a matrix are interchanged then the resultant matrix is called the transpose of the given matrix. It is denoted by A^T (or) A'

$$\begin{aligned}
 \text{Example: If } A &= \begin{bmatrix} 1 & 2 & 3 \\ 4 & -5 & 6 \\ 7 & 0 & 1 \end{bmatrix} \\
 \text{then } A^T &= \begin{bmatrix} 1 & 4 & 7 \\ 2 & -5 & 0 \\ 3 & 6 & 1 \end{bmatrix}
 \end{aligned}$$

Note: (i) If a matrix A is of order $m \times n$, then the order of A^T is $n \times m$.

(ii) $(A^T)^T = A$

Co-factor matrix:

In a matrix, if all the elements are replaced by the corresponding co-factors is called the co-factor matrix.

Example:

The co-factor matrix of the matrix. $\begin{bmatrix} 1 & -4 \\ 8 & 3 \end{bmatrix}$ is as follows

Minors	co-factors
$m_{11} = 3$	$c_{11} = (-1)^{1+1} m_{11} = (-1)^2(3) = 3$
$m_{12} = 8$	$c_{12} = (-1)^{1+2} m_{12} = (-1)^3(8) = -8$
$m_{21} = -4$	$c_{21} = (-1)^{2+1} m_{21} = (-1)^3(-4) = 4$
$m_{22} = 1$	$c_{22} = (-1)^{2+2} m_{22} = (-1)^4(1) = 1$

co-factor matrix is $\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} 3 & -8 \\ 4 & 1 \end{bmatrix}$

Adjoint matrix (or) adjugate matrix:

The transpose of the co-factor matrix is called the adjoint matrix. or adjugate matrix. It is denoted by $\text{adj } A$.

Example

Let $A = \begin{bmatrix} 3 & 2 \\ -3 & 4 \end{bmatrix}$

Cofactor of 3 = $(-1)^{1+1}(4) = 4$

Cofactor of 2 = $(-1)^{1+2}(-3) = 3$

Cofactor of -3 = $(-1)^{2+1}(2) = -2$

Cofactor of 4 = $(-1)^{2+2}(3) = 3$

$$\text{Cofactor matrix} = \begin{bmatrix} 4 & 3 \\ -2 & 3 \end{bmatrix}$$

$$\text{Adj } A = \begin{bmatrix} 4 & -2 \\ 3 & 3 \end{bmatrix}$$

Singular and Non-singular matrices:

A square matrix A is said to be singular if $|A| = 0$. If the determinant value of the square matrix A is not zero it is a non-singular matrix.

Example:

$$\text{Let } A = \begin{bmatrix} 2 & 5 & 7 \\ 7 & 1 & 6 \\ 5 & -4 & -1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & 5 & 7 \\ 7 & 1 & 6 \\ 5 & -4 & -1 \end{vmatrix}$$

$$= 2(-1+24) - 5(-7-30) + 7(-28-5)$$

$$= 46 + 185 - 231 = 0$$

$$\therefore |A| = 0$$

The given matrix A is singular

Inverse of a matrix:

Let A be a non-singular square matrix if there exists a square matrix B , such that $AB=BA=I$ where I is the unit matrix of the same order as that of A , then B is called the inverse of matrix A and it is denoted by A^{-1} . (to be read as A inverse). This can be determined by

using the formula. $A^{-1} = \frac{1}{|A|} \text{adj } A$

Note:

1. if $|A| = 0$, then there is no inverse for the matrix
2. $A^{-1}A = AA^{-1} = I$,
3. $(AB)^{-1} = B^{-1}A^{-1}$
4. $(A^T)^{-1} = (A^{-1})^T$

Working rule to find A^{-1} :

- 1) Find the determinant of A
- 2) Find the co-factor of all elements in A and form the co-factor matrix of A.
- 3) Find the adjoint of A.
- 4) $A^{-1} = \frac{\text{Adj } A}{|A|}$ provided $|A| \neq 0$

Note: For a second order matrix, the adjoint can easily be got by interchanging the principal diagonal elements and changing the signs of the secondary diagonal elements.

Example

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \neq 0$$

$$\text{Adjoint } A = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \therefore A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

WORKED EXAMPLES:

PART – A

1. If $A = \begin{pmatrix} 2 & 1 & -1 \\ 5 & 2 & 3 \end{pmatrix}$ what is the order of the matrix and find A^T

Solution :

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 5 & 2 & 3 \end{pmatrix}$$

The order of the matrix is 2×3

$$A^T = \begin{pmatrix} 2 & 5 \\ 1 & 2 \\ -1 & 3 \end{pmatrix}$$

2. If $A = \begin{pmatrix} 2 & 3 & 0 \\ 5 & 2 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 3 & -1 & 4 \\ 2 & 6 & 7 \end{pmatrix}$ find $3A - 2B$

Solution:

$$\begin{aligned} 3A - 2B &= 3 \begin{pmatrix} 2 & 3 & 0 \\ 5 & 2 & -1 \end{pmatrix} - 2 \begin{pmatrix} 3 & -1 & 4 \\ 2 & 6 & 7 \end{pmatrix} \\ &= \begin{pmatrix} 6 & 9 & 0 \\ 15 & 6 & -3 \end{pmatrix} - \begin{pmatrix} 6 & -2 & 8 \\ 4 & 12 & 14 \end{pmatrix} \\ &= \begin{pmatrix} 6-6 & 9+2 & 0-8 \\ 15-4 & 6-12 & -3-14 \end{pmatrix} = \begin{pmatrix} 0 & 11 & -8 \\ 11 & -6 & -17 \end{pmatrix} \end{aligned}$$

3. If $f(x) = 4x+2$ and $A = \begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix}$ find $f(A)$

Solution:

$$A = \begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix} \text{ and } f(x) = 4x + 2$$

$$f(A) = 4A + 2I_2,$$

$$= 4 \begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix} + 2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 8 & -4 \\ 0 & 12 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$f(A) = \begin{pmatrix} 10 & -4 \\ 0 & 14 \end{pmatrix}$$

4. If $X+Y = \begin{pmatrix} 6 & 4 \\ 2 & 1 \end{pmatrix}$ and $X-Y = \begin{pmatrix} 2 & 4 \\ 8 & 3 \end{pmatrix}$ find X and Y

solution:

Given $X+Y = \begin{pmatrix} 6 & 4 \\ 2 & 1 \end{pmatrix}$ (1)

and $X-Y = \begin{pmatrix} 2 & 4 \\ 8 & 3 \end{pmatrix}$ (2)

Adding $2X = \begin{pmatrix} 8 & 8 \\ 10 & 4 \end{pmatrix} \therefore X = \frac{1}{2} \begin{pmatrix} 8 & 8 \\ 10 & 4 \end{pmatrix} = \begin{pmatrix} 4 & 4 \\ 5 & 2 \end{pmatrix}$

Substitute matrix X in (1)

$$\begin{pmatrix} 4 & 4 \\ 5 & 2 \end{pmatrix} + Y = \begin{pmatrix} 6 & 4 \\ 2 & 1 \end{pmatrix}$$

$$Y = \begin{pmatrix} 6 & 4 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} 4 & 4 \\ 5 & 2 \end{pmatrix}$$

$$Y = \begin{pmatrix} 2 & 0 \\ -3 & -1 \end{pmatrix}$$

5. Find the value of 'a' so that the

matrix $\begin{pmatrix} 1 & -2 & 0 \\ 2 & a & 4 \\ 2 & 1 & 1 \end{pmatrix}$ is singular

solution:

$$\text{Let } A = \begin{pmatrix} 1 & -2 & 0 \\ 2 & a & 4 \\ 2 & 1 & 1 \end{pmatrix}$$

The matrix A is singular, then $|A| = 0$

$$\begin{vmatrix} 1 & -2 & 0 \\ 2 & a & 4 \\ 2 & 1 & 1 \end{vmatrix} = 0$$

Expanding through first row

$$1(a-4) + 2(2-8) = 0$$

$$a-4+4-16 = 0$$

$$a-16 = 0$$

$$a = 16$$

6. If $A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 3 \\ 2 & 4 \end{pmatrix}$ find AB

Solution:

$$\begin{aligned} AB &= \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 0 & 3 \\ 2 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 0+0 & 3+0 \\ 0+2 & 6+4 \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ 2 & 10 \end{pmatrix} \end{aligned}$$

7. Find A^2 , if $A = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$

Solution:

$$\begin{aligned} A^2 &= A.A = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 4+1 & 2+0 \\ 2+0 & 1+0 \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \end{aligned}$$

8. Find the adjoint of $\begin{pmatrix} 3 & 2 \\ -3 & 4 \end{pmatrix}$

Solution:

$$\text{Let } A = \begin{pmatrix} 3 & 2 \\ -3 & 4 \end{pmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$A_{11} = (-1)^{1+1}(4) = (-1)^2 4 = 4$$

$$A_{12} = (-1)^{1+2}(-3) = (-1)^3(-3) = 3$$

$$A_{21} = (-1)^{2+1}(2) = (-1)^3(2) = -2$$

$$A_{22} = (-1)^{2+2}(3) = (-1)^4(3) = 3$$

$$\text{Cofactor matrix } A = \begin{pmatrix} 4 & 3 \\ -2 & 3 \end{pmatrix} \quad \therefore \text{Adj } A = \begin{pmatrix} 4 & -2 \\ 3 & 3 \end{pmatrix}$$

Aliter: Inter Changing elements in the principal diagonal and changing sign of elements in the other diagonal

$$\text{Adj } A = \begin{pmatrix} 4 & -2 \\ 3 & 3 \end{pmatrix}$$

9. Find the inverse of $\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$

Solution:

$$\text{Step 1: let } A = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$$

$$\begin{aligned} \text{Now } |A| &= \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} \\ &= 10 - 12 \\ &= -2 \neq 0 \end{aligned}$$

$\therefore A^{-1}$ exists.

$$\text{adj } A = \begin{pmatrix} 5 & -3 \\ -4 & 2 \end{pmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{Adj } A = \frac{1}{-2} \begin{pmatrix} 5 & -3 \\ -4 & 2 \end{pmatrix}$$

PART-B

$$1) \text{ If } A = \begin{bmatrix} 2 & -1 & 0 \\ 0 & -2 & 1 \\ 1 & 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} -2 & 1 & -1 \\ 1 & 2 & -2 \\ 2 & -1 & -4 \end{bmatrix}$$

Show that $AB = BA$

Solution:

$$\begin{aligned} \text{Now } AB &= \begin{bmatrix} 2 & -1 & 0 \\ 0 & -2 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 & -1 \\ 1 & 2 & -2 \\ 2 & -1 & -4 \end{bmatrix} \\ &= \begin{bmatrix} -4-1+0 & 2-2+0 & -2+2+0 \\ 0-2+2 & 0-4-1 & 0+4-4 \\ -2+0+2 & 1+0-1 & -1+0-4 \end{bmatrix} \end{aligned}$$

$$AB = \begin{bmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$

$$\begin{aligned} BA &= \begin{bmatrix} -2 & 1 & -1 \\ 1 & 2 & -2 \\ 2 & -1 & -4 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ 0 & -2 & 1 \\ 1 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -4+0-1 & 2-2+0 & 0+1-1 \\ 2+0-2 & -1-4+0 & 0+2-2 \\ 4+0-4 & -2+2+0 & 0-1-4 \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$

$$\therefore AB = BA$$

$$2) \text{ Show that } AB \neq BA \text{ if } A = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix} \text{ and } B = \begin{pmatrix} 3 & 2 \\ 1 & -1 \end{pmatrix}$$

Solution:

$$\begin{aligned} \text{Now } AB &= \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 3+2 & 2-2 \\ -3+4 & -2-4 \end{pmatrix} \\ &= \begin{pmatrix} 5 & 0 \\ 1 & -6 \end{pmatrix} \end{aligned}$$

Similarly

$$\begin{aligned} BA &= \begin{pmatrix} 3 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 3-2 & 6+8 \\ 1+1 & 2-4 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 14 \\ 2 & -2 \end{pmatrix} \end{aligned}$$

$$\therefore AB \neq BA$$

$$3) \text{ If } A = \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix} \text{ Find } A^2 + 2A^T + I$$

Solution:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix}$$

Now $A^2 + 2A^T + I$

$$\begin{aligned}
 &= \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix} + 2 \begin{pmatrix} 1 & 3 \\ 2 & -4 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
 &= \begin{pmatrix} 1+6 & 2-8 \\ 3-12 & 6+16 \end{pmatrix} + \begin{pmatrix} 2 & 6 \\ 4 & -8 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
 &= \begin{pmatrix} 7 & -6 \\ -9 & 22 \end{pmatrix} + \begin{pmatrix} 2 & 6 \\ 4 & -8 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
 &= \begin{pmatrix} 10 & 0 \\ -5 & 15 \end{pmatrix}
 \end{aligned}$$

4) If $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$ show that $(A - I)(A - 4I) = 0$

Solution:

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

$$A - I = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$A - 4I = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} - \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$$

$$\begin{aligned}
 (A - I)(A - 4I) &= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \\
 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O
 \end{aligned}$$

5) If $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$ show that $A^2 - 4A = 5I$ and hence find A^3

Solution:

$$\text{Let } A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1+4+4 & 2+2+4 & 2+4+2 \\ 2+2+4 & 4+1+4 & 4+2+2 \\ 2+4+2 & 4+2+2 & 4+4+1 \end{pmatrix}$$

$$= \begin{pmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{pmatrix}$$

$$A^2 - 4A = \begin{pmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{pmatrix} - 4 \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{pmatrix} - \begin{pmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix} = 5 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 5I = \text{RHS}$$

To find A^3

We have proved that $A^2 - 4A = 5I$

$$A^2 = 4A + 5I$$

Multiplying both sides by A, we get

$$A^3 = 4A^2 + 5A = 4A^2 + 5A$$

$$\begin{aligned}
 &= 4 \begin{pmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{pmatrix} + 5 \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix} \\
 &= \begin{pmatrix} 36 & 32 & 32 \\ 32 & 36 & 32 \\ 32 & 32 & 36 \end{pmatrix} + \begin{pmatrix} 5 & 10 & 10 \\ 10 & 5 & 10 \\ 10 & 10 & 5 \end{pmatrix} \\
 &= \begin{pmatrix} 41 & 42 & 42 \\ 42 & 41 & 42 \\ 42 & 42 & 41 \end{pmatrix}
 \end{aligned}$$

6) If $A = \begin{pmatrix} 2 & 3 \\ -1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$ Show that $(AB)^{-1} = B^{-1} A^{-1}$

Solution:

$$\begin{aligned}
 AB &= \begin{pmatrix} 2 & 3 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \\
 &= \begin{pmatrix} -2+0 & 4+3 \\ 1+0 & -2+0 \end{pmatrix} \\
 &= \begin{pmatrix} -2 & 7 \\ 1 & -2 \end{pmatrix} \\
 |AB| &= \begin{vmatrix} -2 & 7 \\ 1 & -2 \end{vmatrix} = 4-7 = -3 \neq 0
 \end{aligned}$$

$(AB)^{-1}$ exists

$$\text{Adj}(AB) = \begin{pmatrix} -2 & -7 \\ -1 & -2 \end{pmatrix}$$

$$(AB)^{-1} = \frac{\text{Adj}(AB)}{|AB|} = \frac{1}{-3} \begin{pmatrix} -2 & -7 \\ -1 & -2 \end{pmatrix} \quad (1)$$

$$B = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$$

$$|B| = \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = -1 \cdot 0 = -1 \neq 0$$

B^{-1} exist

$$\text{Adj}(B) = \begin{pmatrix} 1 & -2 \\ 0 & -1 \end{pmatrix}$$

$$B^{-1} = \frac{\text{Adj}B}{|B|} = \frac{1}{-1} \begin{pmatrix} 1 & -2 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 2 & 3 \\ -1 & 0 \end{pmatrix}$$

$$|A| = \begin{vmatrix} 2 & 3 \\ -1 & 0 \end{vmatrix} = 0 + 3 = 3 \neq 0 \quad A^{-1} \text{ exist}$$

$$\text{Adj} A = \begin{pmatrix} 0 & -3 \\ 1 & 2 \end{pmatrix}$$

$$A^{-1} = \frac{\text{Adj}A}{|A|} = \frac{1}{3} \begin{pmatrix} 0 & -3 \\ 1 & 2 \end{pmatrix}$$

$$B^{-1} A^{-1} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \frac{1}{3} \begin{pmatrix} 0 & -3 \\ 1 & 2 \end{pmatrix}$$

$$= \frac{1}{3} \begin{pmatrix} 1 & -2 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -3 \\ 1 & 2 \end{pmatrix} = -\frac{1}{3} \begin{pmatrix} -2 & -7 \\ -1 & -2 \end{pmatrix} \quad (2)$$

From (1) and (2)

$$(AB)^{-1} = B^{-1}A^{-1}$$

7. Find the inverse of the matrix $\begin{pmatrix} 1 & 1 & -1 \\ 2 & 1 & 0 \\ -1 & 2 & 3 \end{pmatrix}$

Solution :

$$\text{Let } A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 1 & 0 \\ -1 & 2 & 3 \end{pmatrix}$$

$$|A| = \begin{vmatrix} 1 & 1 & -1 \\ 2 & 1 & 0 \\ -1 & 2 & 3 \end{vmatrix}$$

$$= 1(3-0) - 1(6-0) - 1(4+1)$$

$$= 1(3) - 1(6) - 1(5)$$

$$= 3-6-5 = -8 \neq 0 \quad A^{-1} \text{ exists}$$

$$A_{11} = + \begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix} = 3-0=3$$

$$A_{12} = - \begin{vmatrix} 2 & 0 \\ -1 & 3 \end{vmatrix} = - (6-0) = -6$$

$$A_{13} = + \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} = (4+1) = 5$$

$$A_{21} = - \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} = - (3+2) = -5$$

$$A_{22} = + \begin{vmatrix} 1 & -1 \\ -1 & 3 \end{vmatrix} = (3-1) = 2$$

$$A_{23} = - \begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} = - (2+1) = -3$$

$$A_{31} = + \begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} = (0+1) = 1$$

$$A_{32} = - \begin{vmatrix} 1 & -1 \\ 2 & 0 \end{vmatrix} = - (0+2) = -2$$

$$A_{33} = + \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = (1-2) = -1$$

$$\text{Co-factor matrix } A = \begin{pmatrix} 3 & -6 & 5 \\ -5 & 2 & -3 \\ 1 & -2 & -1 \end{pmatrix}$$

$$\text{Adj } A = \begin{pmatrix} 3 & -5 & 1 \\ -6 & 2 & -2 \\ 5 & -3 & -1 \end{pmatrix}$$

$$A^{-1} = \frac{\text{Adj } A}{|A|} = \frac{-1}{8} \begin{pmatrix} 3 & -5 & 1 \\ -6 & 2 & -2 \\ 5 & -3 & -1 \end{pmatrix}$$

EXERCISE

PART-A

- Find the Value of $\begin{vmatrix} x+y & x \\ y+4z & y \end{vmatrix}$
- Find the value of the determinant $\begin{vmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{vmatrix}$
- Find the value of minor 5 in the determinant $\begin{vmatrix} 1 & 0 & -1 \\ 5 & 2 & 4 \\ 3 & -2 & 6 \end{vmatrix}$
- Find the value of 'm' so that $\begin{vmatrix} 2 & -4 & 1 \\ 4 & -2 & -1 \\ 3 & 1 & m \end{vmatrix} = 0$

5. Find the value of 'x' if $\begin{vmatrix} 1 & -1 & 2 \\ 5 & 3 & x \\ 2 & 1 & 4 \end{vmatrix} = 0$

6. Evaluate without expanding $\begin{vmatrix} 18 & 40 & 58 \\ 16 & 36 & 52 \\ 12 & 28 & 40 \end{vmatrix}$

7. Show that $\begin{vmatrix} 1 & x & y+z \\ 1 & y & z+x \\ 1 & z & x+y \end{vmatrix} = 0$

8. Prove that $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$

9. Show that $\begin{vmatrix} 1 & ab & bc+ca \\ 1 & bc & ca+ab \\ 1 & ca & ab+bc \end{vmatrix} = 0$

10. Find the value of $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ 2a & 2b & 2c \end{vmatrix}$

11. Show that $\begin{vmatrix} 4 & -1 & 2 \\ 2 & 3 & 8 \\ 1 & 2 & 5 \end{vmatrix} = 0$

12. If $A = \begin{pmatrix} 1 & 2 \\ 6 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}$ find $2A - B$

13. Find 2x2 matrix A if $a_{ij} = i+j$

14. If $f(x) = x+3$ and $A = \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}$ find $f(A)$

15. If $f(x) = 2x-5$ and $A = \begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix}$ find $f(A)$

16. Show that the matrix $\begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & 3 \\ 2 & -4 & 6 \end{pmatrix}$ is singular
17. Prove that the matrix $\begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$ is non-singular
18. If $A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ find AB
19. If $X = \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix}$ and $Y = \begin{pmatrix} 2 & 0 \\ 1 & 5 \end{pmatrix}$ find XY .
20. If $A = \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}$ find A^2
21. Find the co-factor matrix of $\begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}$
22. Find the adjoint of $\begin{pmatrix} 2 & -1 \\ 6 & 1 \end{pmatrix}$
23. Find the inverse of $\begin{pmatrix} 2 & 3 \\ -1 & 4 \end{pmatrix}$
24. Find the inverse of $\begin{pmatrix} 3 & -1 \\ 0 & 2 \end{pmatrix}$

PART – B

1. Solve by Cramer's rule
 - a. $3x - y + 2z = 8$, $x - y + z = 2$ and $2x + y - z = 1$
 - b. $3x + y + z = 3$, $2x + 2y + 5z = -1$ and $x - 3y - 4z = 2$
 - c. $x + y + z = 3$, $2x + 3y + 4z = 9$ and $3x - y + z = 3$
 - d. $x + y + z = 3$, $2x - y + z = 2$, $3x + 2y - 2z = 3$
 - e. $x + y - z = 4$, $3x - y + z = 4$, $2x - 7y + 3z = -6$
 - f. $x + 2y - z = -3$, $3x + y + z = 4$, $x - y + 2z = 6$

2. Prove that $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

3. Prove that $\begin{vmatrix} x+y & z & z \\ x & y+z & x \\ y & y & z+x \end{vmatrix} = 4xyz$

4. Prove that $\begin{vmatrix} a & bc & abc \\ b & ca & abc \\ c & ab & abc \end{vmatrix} = \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix}$

5. Prove that $\begin{vmatrix} x+a & b & c \\ a & x+b & c \\ a & b & x+c \end{vmatrix} = x^2 (x+a+b+c)$

6. If $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 2 \\ -2 & 2 \end{pmatrix}$ show that $AB = BA$

7. If $A = \begin{pmatrix} 3 & 6 \\ 1 & -2 \end{pmatrix}$ and $B = \begin{pmatrix} 5 & 0 \\ 2 & 3 \end{pmatrix}$ verify that $(AB)^{-1} = B^{-1} A^{-1}$

8. Find the inverse of the following

i) $\begin{pmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{pmatrix}$

ii) $\begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & -2 \\ -1 & -2 & -1 \end{pmatrix}$

iii) $\begin{pmatrix} 1 & 2 & -1 \\ 3 & 8 & 2 \\ 4 & 9 & -1 \end{pmatrix}$

iv) $\begin{pmatrix} 1 & -1 & 1 \\ 2 & 3 & -3 \\ 6 & -2 & -1 \end{pmatrix}$

ANSWERS

PART - A

1 $y^2 - 4xz$

2 1

3 -1

4 $M = -2$

5 $x = 10$

6 0

10 0

12 $\begin{vmatrix} 1 & 2 \\ 13 & -2 \end{vmatrix}$

$\begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix}$

13 $\begin{vmatrix} 5 & 3 \\ 1 & 4 \end{vmatrix}$

14 $\begin{vmatrix} -3 & -2 \\ 0 & -1 \end{vmatrix}$

15 $\begin{vmatrix} 3 & 5 \\ -1 & -2 \end{vmatrix}$

18 $\begin{vmatrix} 1 & -5 \\ 1 & -5 \end{vmatrix}$

19 $\begin{vmatrix} -2 & 0 \\ 5 & 6 \end{vmatrix}$

20 $\begin{vmatrix} 6 & 17 \\ 3 & 1 \end{vmatrix}$

21 $\begin{vmatrix} -1 & 2 \end{vmatrix}$

$$22. \begin{pmatrix} 1 & 1 \\ -6 & 2 \end{pmatrix}$$

$$23. \frac{1}{11} \begin{pmatrix} 4 & -3 \\ 1 & 2 \end{pmatrix}$$

$$24. \frac{1}{6} \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$$

PART - B

1. a) 1,3,4

b) 1,1,-1

c) 1,1,1

d) 1,1,1

e) 2,1,-1

f) 1,-1,2

$$8. \quad i) \frac{1}{20} \begin{pmatrix} 2 & 6 & 4 \\ 21 & -7 & -8 \\ -18 & 6 & 4 \end{pmatrix}$$

$$ii) \frac{-1}{6} \begin{pmatrix} -3 & -1 & -1 \\ 0 & -2 & 4 \\ -3 & 3 & -3 \end{pmatrix}$$

$$iii) \begin{pmatrix} -26 & -7 & 12 \\ 11 & 3 & -5 \\ -5 & -1 & 2 \end{pmatrix}$$

$$iv) \frac{-1}{15} \begin{pmatrix} -9 & -3 & 0 \\ -16 & -7 & 5 \\ -22 & -4 & 5 \end{pmatrix}$$

