LEARNING MATERIAL
ON
DETERMINANT AND MATRICES

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Matrix.

A matrix is a rectangular array of numbers arranged in rows and columns. If there are 'm' rows and 'n' column's in a matrix, it is called an 'm' by 'n' matrix or a matrix of order $m \times n$.

→ The first letter in '$m \times n$' denotes the number of rows and 2nd letter 'n' denotes the number of columns.

Hence
$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cdots & a_{2n} \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

→ If $m = n$ the matrix A is called a square matrix of order $n \times n$ (or simply n). Thus

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}_{n \times n}$$

Types of matrices :-

1) **Row Matrix :-**

A matrix of order $1 \times n$ is called a row matrix.

Exp:- $A = [1 \cdot 2\; 3 \cdot 4\; 5]$

$B = [a, b\; c\; d]$

2) Column matrix:-

A matrix having only one column is called column matrix.

Exp:- $A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{3\times1}$, $B = \begin{bmatrix} a \\ b \end{bmatrix}_{2\times1}$, $C = [C]_{1\times1}$

raw & Column matrix.

3) Square matrix:-

A matrix is said to be square matrix if the no. of raw is equal to no. of column.

Exp:- $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2\times2}$, $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $C = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}_{3\times3}$

4) Diagonal matrix:-

A Square matrix $A = [a_{ij}]_{n\times n}$ is called a diagonal matrix if all the element except those in the main diagonal are zero.

i.e. $a_{ij} = 0, \ \forall i \neq j$.

Exp:- $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

5) Scalar matrix:-

A diagonal matrix is said to be a scalar matrix if all the element of the main diagonal are same.

Exp:- $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

6) **Identity matrix :- / Unit matrix :-**

A Square matrix, $A = [a_{ij}]_{m \times n}$ is said to be identity matrix / Unit matrix, if except main diagonal all the elements are zero and main diagonal elements are 1.

Exp:- $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3 \times 3}$

7) **Null matrix :-**

A matrix is said to be Null matrix if all the element in the matrix are zero.

$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

8) **Upper triangular matrix :- & Lower triangular matrix**

A Square matrix $A = [a_{ij}]_{m \times m}$ is called an upper triangular matrix, if $a_{ij} = 0$, for $\forall\, i > j$

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_2 & a_4 & a_5 \\ a_3 & a_5 & a_6 \end{bmatrix}$$

A square matrix $A = [a_{ij}]_{m \times m}$ is called lower triangular matrix, if $a_{ij} = 0$, $\forall\, i < j$

Exp:- $B = \begin{bmatrix} a_4 & 0 & 0 \\ a_5 & a_7 & 0 \\ a_6 & a_8 & a_9 \end{bmatrix}$

Transpose of matrix :-

$$[A^T]^T = A$$

Equality of Matrices:-

Two matrices A and B are said to be equal if and only if

i) The order of A is equal to the order of B.

ii) Each elements of A is equal to the corresponding elements of B.

Exp:- $\begin{bmatrix} a & b \\ x & y \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ 0 & 6 \end{bmatrix}$

if and only if $a=1$, $b=5$, $x=0$, $y=6$, Algebra of matrix :-

Addition and Subtraction :-

Let A and B are two matrices then A+B & A-B are exist if A and B are same order

Exp:- $A = [a_{ij}]_{m \times n}$ & $B = [b_{ij}]_{m \times n}$

$A+B = [a_{ij}]_{m \times n} + [b_{ij}]_{m \times n}$

$= [a_{ij} + b_{ij}]_{m \times n}$.

$A-B = [a_{ij}]_{m \times n} - [b_{ij}]_{m \times n}$

$= [a_{ij} - b_{ij}]_{m \times n}$.

Exp:- $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2 \times 3}$, $B \begin{bmatrix} -1 & 0 & 1 \\ 4 & 2 & -1 \end{bmatrix}$

$A+B = \begin{bmatrix} 1-1 & 2+0 & 3+1 \\ 4+4 & 5+2 & 6-1 \end{bmatrix}$

$= \begin{bmatrix} 0 & 2 & 4 \\ 8 & 7 & 5 \end{bmatrix}$

Transpose of matrix :-

If $A = [a_{ij}]_{m \times n}$ then $A^T = [a_{ij}]^T_{m \times n} = [a_{ji}]_{n \times m}$

Exp:-

If $A = [a_{ij}]_{m \times n}$ then $A^T = [a_{ij}]^T_{m \times n} = [a_{ji}]_{n \times m}$

Transpose of transpose :-

$[A^T]^T = A$.

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Non-Singular Matrix :-

Transpose of Addition

Properties:-

1) $[A+B]^T = A^T + B^T$

2) $[KA]^T = KA^T$ (K is scalar quantity)

Exp:-

$$2\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2\times1 & 2\times2 \\ 3\times2 & 2\times4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$$

Properties :-

Basic properties [illegible]

1) Additive identity :-

Additive identity

It is evident that $[a_{ij}]_{m\times n} + [0]_{m\times n} = [a_{ij}]_{m\times n}$

and $[a_{ij}]_{m\times n} - [0]_{m\times n} = [a_{ij}]_{m\times n}$,

then $[0]_{m\times n}$ is the additive identity with respect to addition.

~~Exp:-~~ Exp :- $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & -3 \end{bmatrix}_{2\times 3}$ $[0]_{2\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{2\times 3}$

$$A + [0] = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & -3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & -3 \end{bmatrix}_{2\times 3} = A$$

2) Additive inverse :-

The additive inverse of a matrix $A = [a_{ij}]_{m\times n}$ is the matrix $[-a_{ij}]_{m\times n} = -A$,

Exp :- $A = \begin{bmatrix} 5 & -6 \\ 1 & -1 \end{bmatrix}$, $-A = \begin{bmatrix} -5 & 6 \\ -1 & 1 \end{bmatrix}$

$$A + (-A) = \begin{bmatrix} 5 & -6 \\ 1 & -1 \end{bmatrix} + \begin{bmatrix} -5 & 6 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2\times 2}$$

Additive inverse of 1st matrix.

3) Commutative

$A + B = B + A$

Here A & B are same order

Exp:- Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$B = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$

$A + B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} p & q \\ r & s \end{bmatrix}$

$= \begin{bmatrix} a+p & b+q \\ c+r & d+s \end{bmatrix}$

$B + A = \begin{bmatrix} a+p & b+q \\ c+r & d+s \end{bmatrix}$

4) Associative

$(A+B)+C = A+(B+C)$

Here A, B & C are same order.

Matrix multiplication :-

Condition:-

If A and B are two matrices, then we can find AB, if the number of column of A (1st matrix) is equal to the no. of rawe of B (2nd matrix).

Exp:- Let $A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}_{2 \times 3}$

$B = \begin{bmatrix} d & e \\ g & h \\ j & k \end{bmatrix}_{3 \times 2}$

$[AB]_{2\times 2} = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \begin{bmatrix} d & e \\ g & h \\ j & k \end{bmatrix}$

$= \begin{bmatrix} a\times d + b\times g + c\times j & a\times e + b\times h + c\times k \\ d\times d + e\times g + f j & d\times e + e\times h + f\times k \end{bmatrix}$

$= \begin{bmatrix} ad+bg+cj & ae+bh+ck \\ d^2+eg+fj & de+eh+fk \end{bmatrix}$

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Note:-

1) If A and B are two matrices such that $A+B$ and AB are both defined then A and B are both square matrices of the same order.

Condition
- $A+B$ (Order same)
- AB (No. of raw = No. of column)

$$\underset{(p \times p)}{A} + \underset{(p \times p)}{B} \qquad \underset{(p \times p)}{A} \; \underset{(p \times p)}{B}$$

2) If A is an $m \times n$ matrix and if both AB and BA are defined then B is an $n \times m$ matrix

$$\underset{(m \times n)}{A} \; \underset{(n \times m)}{B} \quad \Big| \quad \underset{(n \times m)}{B} \; \underset{(m \times n)}{A}$$

3) Non zero matrices may multiply to a zero matrix.

Exp:- Let $A = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 5 & 6 \end{bmatrix}$

$$AB = \begin{bmatrix} 0+0 & 0+0 \\ 0+0 & 0+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2 \times 2}$$

We know that :-

$ab = 0$, If $a = 0$ or $b = 0$.

4) If $AB = AC$. We can not say as in scalar Algebra, that $B = C$

$$A = \begin{bmatrix} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix}$$

$$AB = \begin{bmatrix} 7 & 10 & 13 \\ 11 & 14 & 17 \\ 15 & 16 & 21 \end{bmatrix}, = AC$$

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Properties of matrix multiplication :- $\boxed{AB = BA}$

1) Matrix multiplication is not commutative in general.

Exp :- $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times2}$, $B = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}_{2\times3}$

AB = find.

BA = is not find.

Or :-

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0-2 & 1+2 \\ 0-4 & 3+4 \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ -4 & 7 \end{bmatrix}$$

$$BA = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 0+3 & 0+4 \\ -1+3 & -2+4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & 2 \end{bmatrix}$$

$$AB \neq BA,$$

Note :-

If I_n is identity matrix then $\boxed{I_n A = A = A I_n}$

2) Matrix multiplication is associative

- If A, B, C are matrices where the product ~~AB, BC~~ are define $A(BC) = (AB)C$

3) Matrix multiplication is distributive with respect to addition

- If A, B, C are the matrices

i) $A(B+C) = AB + AC$

ii) $A(B-C) = AB - AC$.

iii) $(A+B)(C+D) = A(C+D) + B(C+D)$

4) In matrix multiplication a scalar matrix behaves like a scalar multiplier.

Let $K = \begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix}_{2\times 2}$ $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$$KA = \begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} K+0 & 2K+0 \\ 0+3K & 0+4K \end{bmatrix} = \begin{bmatrix} K & 2K \\ 3K & 4K \end{bmatrix}_{2\times 2}$$

$$= K\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

5) The transpose of the product of two matrices is the product of their transpose taken in the reverse order.

$$[AB]^T = B^T A^T$$

$$[ABC]^T = C^T B^T A^T$$

$$[\text{Raw matrix}]^T = [\text{Column matrix}]$$

→ **Orthogonal Matrix :-**

A square matrix A of order 'n' is said to be orthogonal if $AA^T = A^T A = I_n$

Exp :-

→ **Idempotent matrix :-**

A square matrix A is called Idempotent if $A^2 = A \times A = A$

$A = \begin{bmatrix} 4 & -1 \\ 12 & -3 \end{bmatrix}$

Exp :- $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$

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→ **Involuntary matrix :-**

A square matrix such that $\boxed{A^2 = I}$

Exp:- $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ or $B = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$

→ **Nilpotent matrix:-**

A square matrix A is called a nilpotent matrix, if there exist +ve integer 'm' such that

$A^m = A \times A \cdots A \text{ (m times)} = O_{m \times m}$.

Exp:- $A = \begin{bmatrix} 5 & -3 & 2 \\ 15 & -9 & 6 \\ 10 & -6 & 4 \end{bmatrix}$

$$A \times A = \begin{bmatrix} 5 & -3 & 2 \\ 15 & -9 & 6 \\ 10 & -6 & 4 \end{bmatrix} \cdot \begin{bmatrix} 5 & -3 & 2 \\ 15 & -9 & 6 \\ 10 & -6 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 25-45+20 & -15+27-12 & 10-18+8 \\ 75-135+60 & -45+81-36 & 30-54+24 \\ 50-90+40 & -30+54-24 & 20-36+16 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Inverse of a square matrix:-

Adjoint matrix:-

If $A = [a_{ij}]_{n\times n}$ is a square matrix then the transpose of the matrix $[A_{ij}]_{n\times n}$ of which the element are cofactors of the corresponding element in $|A|$.

- It is denoted by $\text{Adj } A = [\text{Cofactor matrix}]^T$

1) Singular matrix:- $|A| = 0$ is called singular matrix.

2) Non singular matrix- Otherwise non-singular matrix

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{32} \end{bmatrix}$

Cofactor matrix:- $\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$ Or $\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$

$= \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$

Find the adjoint of the matrix - $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 1 & 3 & 1 \end{bmatrix}$

Cofactor of $C_{11} = (-1)^{1+1} M_{11}$

$= (-1)^2 \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = 1\times(1-6) = -5$

$C_{12} = (-1)^{1+2} \begin{vmatrix} 2 & 2 \\ 1 & 1 \end{vmatrix} = -1\times 0 = 0$

$C_{13} = 5$

$C_{21} = 2, \quad C_{22} = 0, \quad C_{23} = -2$

$C_{31} = 1, \quad C_{32} = 0, \quad C_{33} = -1$

Cofactor matrix is - $\begin{bmatrix} -5 & 0 & 5 \\ 2 & 0 & -2 \\ 1 & 0 & -1 \end{bmatrix}$

$\text{Adj } A = \begin{bmatrix} -5 & 2 & 1 \\ 0 & 0 & 0 \\ 5 & -2 & -1 \end{bmatrix}$

-: Inverse of the matrix :-

find
Inverse of the matrix $\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$

$$A^{-1} = \frac{\text{adj } A}{|A|}$$

Soln :- $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = 1 - 6 = -5 \neq 0$$

$\therefore$ A^{-1} exists

Now, cofactor of $c_{11} = (-1)^{1+1} \, 1 = 1$

$c_{12} = (-1)^{1+2} \, 3 = -3$

$c_{21} = (-1)^{2+1} \, 2 = -2$

$c_{22} = (-1)^{2+2} \, 1 = 1$

$$\text{cofactor matrix} = \begin{bmatrix} 1 & -3 \\ -2 & 1 \end{bmatrix}$$

$$\text{adj } A = \begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix}$$

Now,
$$A^{-1} = \frac{\text{adj } A}{|A|} = \frac{1}{-5} \begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{5} & \frac{2}{5} \\ \frac{3}{5} & -\frac{1}{5} \end{bmatrix}$$

Solution of a System of Linear Equations by Matrix method:-

Suppose we have following system of equations

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

where $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

The equation in the matrix form:-

$$AX = B$$
$$\Rightarrow X = A^{-1}B$$
$$= \frac{\text{adj } A}{|A|} \cdot B$$

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Determenants

Every Square matrix can be associated to an expression or number which is known as its determents. If $A = [a_{ij}]$ is a square matrix of order 'n' then the det. A is denoted by det. A or $|A|$

Or
$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & a_{n3} \cdots & a_{nn} \end{vmatrix}$$

Determenant of Square matrix of order 1:-

Exp:- $A = [a_{ij}]_{1\times1} = [5]$

$$\det. A = |5| = 5$$

2) Determenant of Square matrix of order '2':-

Exp:- $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$$\det. A = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2$$

3) Order-3.

Let $A = [a_{ij}]_{3\times3}$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$|A| = a_{11}[a_{22}.a_{33} - a_{32}.a_{23}] - a_{12}[a_{21}a_{33} - a_{31}a_{23}] + a_{13}[a_{21}a_{32} - a_{31}a_{22}]$$

Determinant of square matrix of order 3 by using Sarrus Diagram :-

If $A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Sarrus Diagram :-

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{vmatrix}$$

$$= \{(a_{11} \times a_{22} \times a_{33}) + (a_{12} \times a_{23} \times a_{31}) + (a_{13} \times a_{21} \times a_{32})\} - \{(a_{31} \times a_{22} \times a_{13}) + (a_{32} \times a_{23} \times a_{11}) + (a_{33} \times a_{21} \times a_{12})\}$$

(More then '3' Applicable when only 3.)

Singular matrix :-

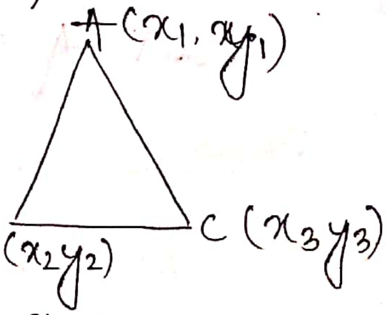
A Square matrix is said to be a singular matrix. If the det. of the square matrix is zero. Other wise it is called Non-singular matrix

Exp:- $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

$\det |A| = 1 \times 4 - 2 \times 2 = 4 - 4 = 0$

$\therefore$ A is a singular matrix.

Area of a Triangle :-



If $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of a triangle ABC. then the Area of Δ ABC :- $|\Delta| = \frac{1}{2} \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix}$

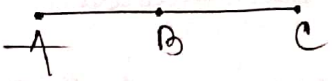
$$= \frac{1}{2}\left[x_1(y_2-y_3) - x_2(y_1-y_3) + x_3(y_1-y_2)\right]$$

$$= \frac{1}{2}\left[x_1(y_2-y_3) + x_2(y_3-y_1) + x_3(y_1-y_2)\right]$$

Condition for co-linearity:-

det. (Area of triangle) = 0

$|\Delta| = 0$



Exp:-

$$A = \begin{vmatrix} 2 & 5 & 7 \\ 3 & 4 & 6 \\ 1 & 1 & 1 \end{vmatrix} \quad \text{Or} \quad \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

Minor & Co-factor:-

Minor:-

Let $A = [a_{ij}]$ be a square matrix of order n, the minor M_{ij} of a_{ij} in A is the determinant of the square submatrix of order $(n-1)$ obtain by leaving ith row & jth column of A.

Exp:- $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$M_{11} = \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} = 45-48 = -3$$

$$M_{12} = \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} = 36-42 = -6$$

$$M_{13} = \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} = 32-35 = -3$$

Co-factor:-

Let $A = [a_{ij}]$ be a square matrix of order n. the cofactor c_{ij} of a_{ij} in A is equal to $(-1)^{i+j} M_{ij}$

$$\boxed{C_{ij} = (-1)^{i+j} M_{ij}}$$

Exp:- $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

$C_{11} = (-1)^{1+1} \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} = 1 \times (45-48) = 1 \times -3 = -3$

$C_{12} = (-1)^{1+2} \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} = (-1) \times (36-42) = (-1) \times (-6) = 6$

$C_{13} = (-1)^{1+3} \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} = 1 \times (32-35) = 1 \times (-3) = -3$

Properties of Determinant:-

1) A determinant remains unaltered/unaltered by changing rows into column and column in rows

Exp:- $\Delta_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ $\Delta_2 = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_2 & c_2 & c_3 \end{vmatrix}$

then $\Delta_1 = \Delta_2$.

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = 4-6 = -2$

$= 4-6 = -2$ $\boxed{A_1 = A_2}$

2) The interchange of two adjacent rows or column of a determinant changes the sign of the determinant without changing its absolute value.

Exp:- $\Delta_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} R_1 \leftrightarrow R_2 \; \Delta_2 = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$\cancel{\Delta_1 = \Delta_2}$

$\rightarrow -\begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$

3) If two rows or two column of a determinant are identical then the value of the determinant is zero.

Exp:- $A = \begin{vmatrix} a_1 & b_1 & c_2 \\ a_1 & b_1 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

4) If every element of any raw or column of a determinant is multiplied by a factor, the determinant is multiplied by the same factor.

Exp:- $A = \begin{vmatrix} Ka_1 & Kb_1 & Kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = K \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Exp - $A = \begin{vmatrix} 2 & 4 \\ 1 & 5 \end{vmatrix}$ Or $A = 2\begin{vmatrix} 1 & 2 \\ 1 & 5 \end{vmatrix}$

$= 10 - 4 = 6$ $= 2(5-2) = 2 \times 3 = 6$

5) If every element of a raw or a column of a det. can be expressed as the sum of two numbers, then the det. can be expressed as the sum of two determinants.

Exp:- $\begin{vmatrix} a_1+\alpha_1 & b_1 & c_1 \\ a_2+\alpha_2 & b_2 & c_2 \\ a_3+\alpha_3 & b_3 & c_3 \end{vmatrix}$

$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & b_1 & c_1 \\ \alpha_2 & b_2 & c_2 \\ \alpha_3 & b_3 & c_3 \end{vmatrix}$

6) If $A = [a_{ij}]$ is a diagonal matrix of order $(n \geq 2)$ then $|A| = a_{11} \times a_{22} \times a_{33} \dots a_{nn}$

Exp:- $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = 1 \times 2 \times 3 = 6$

7) If $A = [a_{ij}]$ is a Upper triangular Or Lower triangular det. then the $|A| = a \times b \times c \dots n$

Exp:- $\begin{bmatrix} a & d & e \\ 0 & b & f \\ 0 & 0 & c \end{bmatrix}$ Or $\begin{vmatrix} a & 0 & 0 \\ d & b & 0 \\ e & f & c \end{vmatrix}$

$|A| = a \times b \times c$ $|A| = a \times b \times c$.

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8) A determinant remains unchanged by adding K times the elements of any raw or column to the corresponding elements of any other raw or column, where K is any given number.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad R_2 \to R_2 + KR_3$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2+Ka_3 & b_2+Kb_3 & c_2+Kc_3 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

9) If A and B are the matrices of same order then $|AB| = |A||B|$

Exp:- $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}$

$AB = \begin{bmatrix} 3+2 & 0+4 \\ 9+4 & 0+8 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 13 & 8 \end{bmatrix}$

$|A| = 4-6 = -2$

$|B| = 6-0 = 6$

$|AB| = 40 - 52 = -12$

$|A||B| = -2 \times 6 = -12$

10) The sum of the products of the elements of any raw or column of a determinant and the cofactor of the corresponding elements of any other raw or column of the determinant is zero.

Exp:- $A = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 C_{11} + b_1 C_{12} + c_1 C_{13} = |A|$

$$a_1 C_{21} + b_1 C_{22} + c_1 C_{23} = 0$$

Some special types of Determinant :-

1) Symmetric Determinant:- It is a det. with $\boxed{a_{ij} = a_{ji}}$.

Exp:- $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = abc + 2fgh - af^2 - bg^2 - ch^2$

2) Skew Symmetric Determinant:- $\boxed{a_{ij} = -a_{ji}}$

$$\begin{vmatrix} 0 & b & -c \\ -b & 0 & d \\ c & -d & 0 \end{vmatrix} = 0$$

-: Inverse of the matrix :-

find Inverse of the matrix $\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$ $\qquad A^{-1} = \frac{\text{adj } A}{|A|}$

Solⁿ:- $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = 1-6 = -5 \neq 0$$

$\therefore$ A^{-1} exists

Now, cofactor of $c_{11} = (-1)^{1+1}\, 1 = 1$

$c_{12} = (-1)^{1+2}\, 3 = -3$

$c_{21} = (-1)^{2+1}\, 2 = -2$

$c_{22} = (-1)^{2+2}\, 1 = 1$

cofactor matrix $= \begin{bmatrix} 1 & -3 \\ -2 & 1 \end{bmatrix}$

$\text{adj } A = \begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix}$

Now, $A^{-1} = \frac{\text{adj } A}{|A|} = \frac{1}{-5}\begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix}$

$$= \begin{bmatrix} -\frac{1}{5} & \frac{2}{5} \\ \frac{3}{5} & -\frac{1}{5} \end{bmatrix}$$

Exp-3 If $x+y+z=0$. Show that $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^3 & y^3 & z^3 \end{vmatrix} = 0$

Solⁿ:- $x+y+z=0$.

$\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^3 & y^3 & z^3 \end{vmatrix}$ $\quad C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$

$$= \begin{vmatrix} 1 & 0 & 0 \\ x & y-x & z-x \\ x^3 & y^3-x^3 & z^3-x^3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ x & (y-x) & z-x \\ x^3 & (y-x)(y^2+xy+x^2) & (z-x)(z^2+xz+x^2) \end{vmatrix}$$

$$= (y-x)(z-x)\begin{vmatrix} 1 & 0 & 0 \\ x & 1 & 1 \\ x^3 & y^2+xy+x^2 & z^2+xz+x^2 \end{vmatrix}$$

$$= (y-x)(z-x)\left[z^2+xz+\cancel{x^2}-y^2-xy-\cancel{x^2}\right]$$

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$$= (y-x)(z-x)\left[x(z-y) + (z+y)(z-y)\right]$$
$$= (y-x)(z-x)(z-y)(x+y+z)$$
$$= (y-x)(z-x)(z-y)\times 0 = 0$$

Exp-4 Proove that $(a-1)$ is a factor of the det. $\begin{vmatrix} a+1 & 2 & 3 \\ 3 & a+2 & 4 \\ 4 & 4 & a+3 \end{vmatrix}$

Soln:-
$$\begin{vmatrix} a+1 & 2 & 3 \\ 3 & a+2 & 4 \\ 4 & 4 & a+3 \end{vmatrix} \quad C_1 \to C_1 - C_2$$

$$= \begin{vmatrix} a+1-2 & 2 & 3 \\ 3-a-2 & a+2 & 4 \\ 4-4 & 4 & a+3 \end{vmatrix}$$

$$= \begin{vmatrix} a-1 & 2 & 3 \\ 1-a & a+2 & 4 \\ 0 & 4 & a+3 \end{vmatrix}$$

$$= \begin{vmatrix} a-1 & 2 & 3 \\ -(a-1) & a+2 & 4 \\ 0 & 4 & a+3 \end{vmatrix} = (a-1)\begin{vmatrix} 1 & 2 & 3 \\ -1 & a+2 & 4 \\ 0 & 4 & a+3 \end{vmatrix}$$

$$= (a-1)\left[(a+2)(a+3) - 4 + 2(2(a+3) - 12)\right]$$
$$= (a-1)\left[(a+2)(a+3) - 4 + 2(a+3) - 12\right]$$

Cramer's rule :-

Let us consider the linear equation —

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \quad \Delta x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}, \quad \Delta y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$$

$$\Delta z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

$$x = \frac{\Delta x}{\Delta}, \quad y = \frac{\Delta y}{\Delta}, \quad z = \frac{\Delta z}{\Delta}$$

Note :-

1) Cramer's rule is not applicable when $\Delta = 0$.
2) If $\Delta \neq 0$, $\Delta_1 = \Delta_2 = \Delta_3 = 0$, then the only solution of equation
3) If $\Delta = 0$, but at least one of $\Delta_1, \Delta_2, \Delta_3$ is not zero then the system has no solution.
4) If $\Delta = \Delta_1 = \Delta_2 = \Delta_3 = 0$, the system has infinite number of solution.

Invertible matrix :-

If A is square matrix of order 'n' and if there exist 'B' also a square matrix of order n such that $AB = BA = I_n$, then 'B' is called inverse matrix of A. It is denoted by A^{-1}. In that case A is invertible.

Exp :- $A = \begin{bmatrix} 2 & 3 \\ 2 & 2 \end{bmatrix}$, $B = \begin{bmatrix} -1 & \frac{3}{2} \\ 1 & -1 \end{bmatrix}$

$\therefore \boxed{AB = BA = I_2}$

→ If the matrix is invertible if $\det A \neq 0$.

Exp :- $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 2 & 4 & 6 \end{bmatrix}$

$$|A| = 1(24-20) - 2(18-10) + 3(12-8)$$
$$= 4 - 16 + 12$$
$$= 16 - 16 = 0$$

A is not invertible.