

LEARNING MATERIAL ON INTEGRATION

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INTEGRATION

Indefinite Integration:-

Definition:-

Integration is the reverse process of differentiation. It is also known as primitive or antiderivative of the given function.

If the differential coefficient of $F(x)$ is $f(x)$ i.e. $\frac{d}{dx}(F(x)) = f(x)$, then we say that the antiderivative or integral of $f(x)$ is $F(x)$ and is denoted by -

$$\int f(x) dx = F(x) + C$$

where 'C' be the integration arbitrary constant.

For example, let $F(x) = x^2$, then

$$\frac{d}{dx}(x^2) = 2x \Rightarrow \int 2x dx = x^2$$

Again, let $F(x) = x^2 + a$, then

$$\frac{d}{dx}(x^2 + a) = 2x \Rightarrow \int \cancel{2x} dx = x^2 + a$$

Here $x^2 = x^2 \Rightarrow \int 2x dx = \int \cancel{2x} dx$

But $x^2 \neq x^2 + a$

Hence we always add the arbitrary integration constant.

$$\text{Hence, } \frac{d}{dx}(x^2) = 2x \Rightarrow \int 2x dx = x^2 + C$$

$$\frac{d}{dx}(x^2 + a) = 2x \Rightarrow \int 2x dx = x^2 + C$$

Hence if $\frac{d}{dx} f(x) = g(x)$

$$\Rightarrow \int g(x) dx = f(x) + K$$

If $g(x)$ be the derivative of $f(x)$, then $f(x) + K$ be the indefinite integral of $g(x)$.

Integration Formulae :-

$$* \int x^n dx = \frac{x^{n+1}}{n+1} + K, \quad n \neq -1$$

$$* \int \frac{1}{x} dx = \int x^{-1} dx = \ln|x| + K$$

$$* \int e^x dx = e^x + K$$

$$* \int a^x dx = \frac{a^x}{\ln a} + K$$

$$* \int \sin x dx = -\cos x + K$$

$$* \int \cos x dx = \sin x + K$$

$$* \int \sec^2 x dx = \tan x + K$$

$$* \int \operatorname{cosec}^2 x dx = -\cot x + K$$

$$* \int \sec x \cdot \tan x dx = \sec x + K$$

$$* \int \operatorname{cosec} x \cdot \cot x dx = -\operatorname{cosec} x + K$$

$$* \int \tan x dx = \ln|\sec x| + K = -\log|\cos x| + K$$

$$* \int \cot x dx = \ln|\sin x| + K =$$

$$* \int \sec x dx = \ln|\sec x + \tan x| + K$$

$$* \int \operatorname{cosec} x dx = \ln|\operatorname{cosec} x - \cot x| + K$$

$$* \int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + K \quad \text{or} \quad -\cos^{-1} x + K$$

$$* \int \frac{1}{1+x^2} dx = \tan^{-1} x + K \quad \text{or} \quad -\cot^{-1} x + K$$

$$* \int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x + K \quad \text{or} \quad -\operatorname{cosec}^{-1} x + K$$

Algebra of Integrals:

$$* \int [f(x) \pm g(x)] dx = \int f(x) dx \pm \int g(x) dx$$

$$* \int \lambda f(x) dx = \lambda \int f(x) dx, \text{ for a constant } \lambda.$$

$$\text{Thus, } \int [\lambda_1 f_1(x) \pm \lambda_2 f_2(x) \pm \dots \pm \lambda_n f_n(x)] dx \\ = \lambda_1 \int f_1(x) dx \pm \lambda_2 \int f_2(x) dx \pm \dots \pm \lambda_n \int f_n(x) dx.$$

eg:- * $I = \int (x^6 + x^2 + x + 1) dx$

$$\Rightarrow I = \int x^6 dx + \int x^2 dx + \int x dx + \int dx$$

$$= \frac{x^{6+1}}{6+1} + \frac{x^{2+1}}{2+1} + \frac{x^{1+1}}{1+1} + x$$

$$= \frac{x^7}{7} + \frac{x^3}{3} + \frac{x^2}{2} + x + K$$

* $I = \int 6x^3 (x+5)^2 dx$

$$\Rightarrow I = \int 6x^3 (x^2 + 25 + 10x) dx$$

$$= \int (6x^5 + 150x^3 + 60x^4) dx$$

$$= 6 \int x^5 dx + 150 \int x^3 dx + 60 \int x^4 dx$$

$$= 6 \cdot \frac{x^6}{6} + 150 \cdot \frac{x^4}{4} + 60 \cdot \frac{x^5}{5} + K$$

$$= x^6 + \frac{75}{2} x^4 + 12x^5 + K$$

$$\Rightarrow I = x^6 + 12x^5 + \frac{75}{2} x^4 + K$$

* $I = \int \left(4 \cos x - 3e^x + \frac{2}{\sqrt{1-x^2}} \right) dx$

$$\Rightarrow I = 4 \int \cos x dx - 3 \int e^x dx + 2 \int \frac{dx}{\sqrt{1-x^2}}$$

$$= 4 \sin x - 3e^x + 2 \sin^{-1} x + K$$

* $I = \int 5 \tan^2 x dx$

$$\Rightarrow I = 5 \int \tan^2 x dx = 5 \int (\sec^2 x - 1) dx$$

$$= 5 \left[\int \sec^2 x \, dx - \int dx \right]$$

$$= 5(\tan x - x) + K$$

$$* I = \int \frac{x^4}{x^2+1} \, dx$$

$$= \int \frac{x^4 - 1 + 1}{x^2+1} \, dx$$

$$= \int \left[\frac{x^4-1}{x^2+1} + \frac{1}{x^2+1} \right] dx$$

$$= \int \left[\frac{(x^2+1)(x^2-1)}{x^2+1} + \frac{1}{x^2+1} \right] dx$$

$$= \int \left(x^2 - 1 + \frac{1}{x^2+1} \right) dx$$

$$= \int x^2 \, dx - \int dx + \int \frac{dx}{1+x^2}$$

$$= \frac{x^3}{3} - x + \tan^{-1} x + K$$

$$* I = \int \frac{(x+\sqrt{x})(2x+1)}{x^2} \, dx$$

$$= \int \left(\frac{x+\sqrt{x}}{x} \times \frac{2x+1}{x} \right) dx$$

$$= \int \left[(1+x^{-1/2})(2+x^{-1}) \right] dx$$

$$= \int (2 + 2x^{-1/2} + x^{-1} + x^{-3/2}) dx$$

$$= 2x + \frac{2x^{-1/2+1}}{-1/2+1} + \ln x + \frac{x^{-3/2+1}}{-3/2+1}$$

$$= 2x + \frac{2x^{1/2}}{1/2} + \ln x +$$

$$\frac{x^{-1/2}}{-1/2}$$

$$= 2x + 4\sqrt{x} + \ln x - \frac{2}{\sqrt{x}} + K$$

$$* I = \int e^{3x} \, dx$$

$$= \int (e^3)^x \, dx$$

$$= \frac{(e^3)^x}{\ln e^3} + K$$

$$= \frac{e^{3x}}{3} + K$$

$$* I = \int \frac{3x^2}{x^2+1} \, dx$$

$$= \int \frac{3x^2+3-3}{x^2+1} \, dx$$

$$= \int \frac{3(x^2+1)-3}{x^2+1} \, dx$$

$$= \int \left(3 - \frac{3}{x^2+1} \right) dx$$

$$= 3x - 3 \tan^{-1} x + K$$

$$= 3(x - \tan^{-1} x) + K$$

$$* I = \int \left(x^{4/7} + \frac{1}{x^{1/3}} \right) dx$$

$$= \int x^{4/7} \, dx + \int x^{-1/3} \, dx$$

$$= \frac{x^{4/7+1}}{\frac{4}{7}+1} + \frac{x^{-1/3+1}}{-\frac{1}{3}+1} + K$$

$$= \frac{7}{11} x^{11/7} + \frac{3}{2} x^{2/3} + K$$

$$\begin{aligned}
 * I &= \int \frac{\sin^2 x}{1 + \cos x} dx \\
 &= \int \frac{1 - \cos^2 x}{1 + \cos x} dx \\
 &= \int \frac{(1 - \cos x)(1 + \cos x)}{(1 + \cos x)} dx \\
 &= \int (1 - \cos x) dx \\
 &= x - \sin x + k
 \end{aligned}$$

$$\begin{aligned}
 * I &= \int \sec^2 x \cdot \operatorname{cosec}^2 x dx \\
 &= \int \left(\frac{1}{\cos^2 x} \cdot \frac{1}{\sin^2 x} \right) dx \\
 &= \int \frac{1}{\sin^2 x \cdot \cos^2 x} dx \\
 &= \int \left(\frac{\sin^2 x + \cos^2 x}{\sin^2 x \cdot \cos^2 x} \right) dx \\
 &= \int \left(\frac{1}{\cos^2 x} + \frac{1}{\sin^2 x} \right) dx \\
 &= \int (\sec^2 x + \operatorname{cosec}^2 x) dx \\
 &= \tan x - \cot x + k
 \end{aligned}$$

$$\begin{aligned}
 * I &= \int \sqrt{1 - \cos 2x} dx \\
 &= \int \sqrt{2 \sin^2 x} dx \\
 &= \sqrt{2} \int \sin x dx \\
 &= -\sqrt{2} \cos x + k
 \end{aligned}$$

Integration by substitution:-

By the concept of chain rule,

$$\frac{d}{dx} \{F(g(x)) + k\} = \frac{d}{d\theta} [F(\theta) + k] \cdot \frac{d\theta}{dx}$$

where $\theta = g(x)$

$$\therefore \frac{d}{dx} \{F(g(x)) + k\} = f(\theta) \cdot \frac{d\theta}{dx} = f(\theta) \cdot g'(x)$$

$$\Rightarrow \boxed{\frac{d}{dx} \{F(g(x)) + k\} = f\{g(x)\} g'(x)}$$

$$\text{Now, } I = \int f(g(x)) \cdot g'(x) dx$$

$$\text{Put } \theta = g(x) \Rightarrow \frac{d\theta}{dx} = g'(x) \Rightarrow \boxed{d\theta = g'(x) dx}$$

$$\therefore I = \int f(\theta) d\theta = F(\theta) + k$$

$$\Rightarrow I = F(g(x)) + k$$

$$\text{Hence, } \boxed{\int f(g(x)) \cdot g'(x) dx = F(g(x)) + k}$$

$$\text{Ex:- } * I = \int (ax+b)^n dx, n \neq -1$$

$$\text{Put } \theta = ax+b \Rightarrow \frac{d\theta}{dx} = a \Rightarrow d\theta = a dx \Rightarrow dx = \frac{1}{a} d\theta$$

$$\therefore I = \int \theta^n \frac{d\theta}{a} = \frac{1}{a} \int \theta^n d\theta = \frac{1}{a} \frac{\theta^{n+1}}{n+1} + k$$

$$\Rightarrow I = \frac{1}{a} \frac{(ax+b)^{n+1}}{n+1} + k$$

$$\therefore \int (ax+b)^n dx = \frac{1}{a} \frac{(ax+b)^{n+1}}{n+1} + k$$

$$* \int \cos(ax+b) dx = I$$

$$\text{Put } \theta = ax+b \Rightarrow d\theta = \frac{d\theta}{a}$$

$$\therefore I = \int \frac{1}{a} \cos \theta d\theta = \frac{1}{a} \int \cos \theta d\theta = \frac{1}{a} \sin \theta + k$$

$$\therefore \int \cos(ax+b) dx = \frac{1}{a} \sin(ax+b) + K$$

$$\text{Similarly, } \int \sin(ax+b) dx = -\frac{1}{a} \cos(ax+b) + K$$

$$\int \sec^2(ax+b) dx = \frac{1}{a} \tan(ax+b) + K$$

$$\int \operatorname{cosec}^2(ax+b) dx = -\frac{1}{a} \cot(ax+b) + K$$

Hence, if $\int f(x) dx = g(x)$ then

$$\boxed{\int f(ax+b) dx = \frac{1}{a} g(ax+b)}$$

$$* \int \frac{g'(x)}{g(x)} dx = I$$

$$\text{Put } \theta = g(x) \Rightarrow d\theta = g'(x) dx$$

$$\therefore I = \int \frac{d\theta}{\theta} = \ln \theta = \ln |g(x)| + K$$

Formulae :-

$$\bullet \int \frac{dx}{ax+b} = \frac{1}{a} \int \frac{a dx}{ax+b} = \frac{1}{a} \ln(ax+b) + K$$

$$\bullet \int \cot x dx = \int \frac{\cos x}{\sin x} dx = \int \frac{d(\sin x)}{\sin x} = \ln |\sin x| + K$$

$$[\because \theta = \sin x \Rightarrow d\theta = \cos x \cdot dx]$$

$$\bullet \int \tan x dx = \int \frac{\sin x}{\cos x} dx = -\int \frac{d(\cos x)}{\cos x} = -\ln |\cos x| + K \\ = \ln |\sec x| + K$$

$$\bullet \int \sec x dx = \int \frac{\sec x (\sec x + \tan x)}{\sec x + \tan x} dx \\ = \int \left(\frac{\sec^2 x + \sec x \cdot \tan x}{\sec x + \tan x} \right) dx$$

$$\text{Put } \theta = \sec \alpha + \tan \alpha$$

$$\Rightarrow d\theta = (\sec \alpha \cdot \tan \alpha + \sec^2 \alpha) d\alpha$$

$$\therefore I = \int \frac{d\theta}{\theta} = \ln|\theta| + K = \ln|\sec \alpha + \tan \alpha| + K$$

$$\begin{aligned} \Rightarrow \int \sec \alpha d\alpha &= \ln|\sec \alpha + \tan \alpha| + K \\ &= \ln\left|\tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right)\right| + K \end{aligned}$$

$$\begin{aligned} * \int \operatorname{cosec} \alpha d\alpha &= \int \frac{\operatorname{cosec} \alpha (\operatorname{cosec} \alpha - \cot \alpha)}{(\operatorname{cosec} \alpha - \cot \alpha)} d\alpha \\ &= \int \frac{(\operatorname{cosec}^2 \alpha - \operatorname{cosec} \alpha \cdot \cot \alpha)}{\operatorname{cosec} \alpha - \cot \alpha} d\alpha \end{aligned}$$

$$\text{Put } \theta = \operatorname{cosec} \alpha - \cot \alpha$$

$$\Rightarrow d\theta = (-\operatorname{cosec} \alpha \cdot \cot \alpha + \operatorname{cosec}^2 \alpha) d\alpha$$

$$\therefore I = \int \frac{d\theta}{\theta} = \ln|\theta| + K = \ln|\operatorname{cosec} \alpha - \cot \alpha| + K$$

$$\begin{aligned} \Rightarrow \int \operatorname{cosec} \alpha d\alpha &= \ln|\operatorname{cosec} \alpha - \cot \alpha| + K \\ &= \ln\left|\tan \frac{\alpha}{2}\right| + K \end{aligned}$$

$$\underline{\text{eg:}} \quad I = \int \frac{x^4 + 4x^3}{x^5 + 5x^4 + 7} dx$$

$$\text{Put } \theta = x^5 + 5x^4 + 7 \Rightarrow \frac{d\theta}{dx} = 5x^4 + 20x^3$$

$$\Rightarrow d\theta = (5x^4 + 20x^3) dx$$

$$\Rightarrow \frac{1}{5} d\theta = (x^4 + 4x^3) dx$$

$$\therefore I = \int \frac{1}{\theta} \cdot \frac{d\theta}{5} = \frac{1}{5} \int \frac{d\theta}{\theta} = \frac{1}{5} \ln|\theta| + K$$

$$\Rightarrow I = \frac{1}{5} \ln|x^5 + 5x^4 + 7| + K$$

$$\bullet I = \int 2e^{\tan^2 x} \tan x \cdot \sec^2 x \, dx$$

$$\text{Put } \theta = \tan^2 x$$

$$\Rightarrow d\theta = 2 \tan x \cdot \sec^2 x \, dx$$

$$\therefore I = \int e^{\tan^2 x} (2 \tan x \cdot \sec^2 x) \, dx$$

$$= \int e^\theta \, d\theta = e^\theta + k = e^{\tan^2 x} + k$$

$$\bullet I = \int \frac{(\tan^{-1} x)^3}{1+x^2} \, dx$$

$$\text{Put } \theta = \tan^{-1} x \Rightarrow \frac{d\theta}{dx} = \frac{1}{1+x^2} \Rightarrow d\theta = \frac{dx}{1+x^2}$$

$$\therefore I = \int \theta^3 \cdot d\theta = \frac{\theta^4}{4} + k = \frac{1}{4} (\tan^{-1} x)^4 + k$$

$$\bullet I = \int \frac{3(\ln x)^2}{x} \, dx$$

$$\text{Put } \theta = \ln x \Rightarrow \frac{d\theta}{dx} = \frac{1}{x} \Rightarrow d\theta = \frac{dx}{x}$$

$$\therefore I = \int 3\theta^2 \cdot d\theta = 3 \frac{\theta^3}{3} + k = \theta^3 + k$$

$$\Rightarrow I = (\ln x)^3 + k$$

$$\bullet I = \int x^2 \sin x^3 \, dx$$

$$\text{Put } t = x^3 \Rightarrow \frac{dt}{dx} = 3x^2 \Rightarrow dt = 3x^2 \, dx$$

$$\therefore I = \frac{1}{3} \int \sin t \cdot 3x^2 \, dx$$

$$= \frac{1}{3} \int \sin t \, dt$$

$$= \frac{1}{3} (-\cos t) + k = -\frac{1}{3} \cos x^3 + k$$

$$\bullet I = \int \sin 3x \cdot \sin 6x \, dx$$

$$= \int \sin 3x \cdot (2 \sin 3x \cdot \cos 3x) \, dx$$

$$= \int 2 \sin^2 3x \cdot \cos 3x \, dx$$

$$\text{Put } t = \sin 3x \Rightarrow dt = 3 \cos 3x \, dx$$

$$\therefore I = \frac{2}{3} \int \sin^2 3x \cdot 3 \cos 3x dx$$

$$= \frac{2}{3} \int t^2 dt$$

$$= \frac{2}{3} \times \frac{t^3}{3} + K = \frac{2}{9} \sin^3 3x + K$$

$$\bullet I = \int \frac{dx}{\sqrt{x}(2-\sqrt{x})}$$

Put $t = 2 - \sqrt{x} \Rightarrow \frac{dt}{dx} = -\frac{1}{2\sqrt{x}} \Rightarrow -2dt = \frac{dx}{\sqrt{x}}$

$$\therefore I = \int \frac{-2dt}{t} = -2 \ln|t| + K = -2 \ln|2 - \sqrt{x}| + K$$

$$\bullet I = \int \frac{x^2 dx}{(a+bx)^2}$$

Put $t = (a+bx) \Rightarrow \frac{dt}{dx} = b \Rightarrow \frac{dt}{b} = dx$

Again, $bx = t - a \Rightarrow x = \left(\frac{t-a}{b}\right)^2$

$$\therefore I = \int \frac{\left(\frac{t-a}{b}\right)^2}{t^2} \frac{dt}{b}$$

$$= \frac{1}{b} \int \left(\frac{t^2 - 2at + a^2}{b^2 t^2} \right) dt$$

$$= \frac{1}{b^3} \int \left(\frac{t^2 - 2at + a^2}{t^2} \right) dt$$

$$= \frac{1}{b^3} \int \left(1 - \frac{2a}{t} + \frac{a^2}{t^2} \right) dt$$

$$= \frac{1}{b^3} \left[t - 2a \ln t - \frac{a^2}{t} \right] + K$$

$$= \frac{1}{b^3} \left[(a+bx) - 2a \ln(a+bx) - \frac{a^2}{(a+bx)} \right] + K$$

Integration of some Trigonometric Functions:

$$\begin{aligned} * \sin m\alpha \cdot \cos n\alpha &= \frac{1}{2} [2 \sin m\alpha \cdot \cos n\alpha] \\ &= \frac{1}{2} [\sin (m+n)\alpha + \sin (m-n)\alpha] \end{aligned}$$

$$* \sin m\alpha \cdot \sin n\alpha = \frac{1}{2} [\cos (m-n)\alpha - \cos (m+n)\alpha]$$

$$* \cos m\alpha \cdot \cos n\alpha = \frac{1}{2} [\cos (m-n)\alpha + \cos (m+n)\alpha]$$

e.g.:- $I = \int \sin 3\alpha \cdot \cos 2\alpha \, d\alpha$

$$= \frac{1}{2} \int 2 \sin 3\alpha \cdot \cos 2\alpha \, d\alpha$$

$$= \frac{1}{2} \int [\sin (3+2)\alpha + \sin (3-2)\alpha] \, d\alpha$$

$$= \frac{1}{2} \int (\sin 5\alpha + \sin \alpha) \, d\alpha$$

$$= \frac{1}{2} \left[-\frac{1}{5} \cos 5\alpha - \cos \alpha \right] + K$$

$$= -\frac{1}{10} (\cos 5\alpha + 5 \cos \alpha) + K$$

• $I = \int \sin 2\alpha \cdot \sin \alpha \, d\alpha$

$$= \frac{1}{2} \int 2 \sin 2\alpha \cdot \sin \alpha \, d\alpha$$

$$= \frac{1}{2} \int (\cos \alpha - \cos 3\alpha) \, d\alpha$$

$$= \frac{1}{2} \left(\sin \alpha - \frac{1}{3} \sin 3\alpha \right) + K$$

$$= \frac{1}{6} (3 \sin \alpha - \sin 3\alpha) + K$$

• $I = \int \sin^2 \alpha \, d\alpha$

$$= \int \frac{1}{2} (1 - \cos 2\alpha) \, d\alpha$$

$$= \frac{1}{2} \int (1 - \cos 2\alpha) \, d\alpha$$

$$= \frac{1}{2} \left(\alpha - \frac{1}{2} \sin 2\alpha \right) + K = \frac{1}{4} \left(2\alpha - \sin 2\alpha \right) + K$$

$$\cdot I = \int \cos^3 x \, dx$$

we know $\cos 3x = -3\cos x + 4\cos^3 x$

$$\Rightarrow 4\cos^3 x = 3\cos x + \cos 3x$$

$$\Rightarrow \cos^3 x = \frac{1}{4}(3\cos x + \cos 3x)$$

$$\therefore I = \int \frac{1}{4}(3\cos x + \cos 3x) \, dx$$

$$= \frac{1}{4} \int (3\cos x + \cos 3x) \, dx$$

$$= \frac{1}{4} \left(3\sin x + \frac{1}{3}\sin 3x \right) + k$$

$$= \frac{1}{12} (9\sin x + \sin 3x) + k$$

Hence,

$$\cdot \sin^2 x = \frac{1}{2} \cdot 2\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$\cdot \cos^2 x = \frac{1}{2} \cdot 2\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

$$\cdot \cos^3 x = \frac{1}{4} \cdot 4\cos^3 x = \frac{1}{4}(3\cos x + \cos 3x)$$

$$\cdot \sin^3 x = \frac{1}{4} \cdot 4\sin^3 x = \frac{1}{4}(3\sin x - \sin 3x)$$

$$\cdot \cos^4 x = \left(\frac{2\cos^2 x}{2} \right)^2 = \left(\frac{1 + \cos 2x}{2} \right)^2$$

$$= \frac{1 + \cos^2 2x + 2\cos 2x}{4}$$

$$= \frac{1}{4} \left(1 + \frac{1}{2}(1 + \cos 4x) + 2\cos 2x \right)$$

$$= \frac{1}{8} (2 + 1 + \cos 4x + 4\cos 2x)$$

$$= \frac{1}{8} (3 + 4\cos 2x + \cos 4x)$$

$$\cdot \sin^4 x = \left(\frac{2\sin^2 x}{2} \right)^2 = \left(\frac{1 - \cos 2x}{2} \right)^2$$

$$= \frac{1}{4} (1 - 2\cos 2x + \cos^2 2x)$$

$$= \frac{1}{4} \left[1 - 2\cos 2x + (1 + \cos 2x) \cdot \frac{1}{2} \right]$$

$$= \frac{1}{8} (3 - 4\cos 2x + \cos 4x)$$

$$\begin{aligned}
 \bullet \sin^2 \alpha \cdot \cos^2 \alpha &= \frac{1}{4} \cdot 4 \sin^2 \alpha \cdot \cos^2 \alpha \\
 &= \frac{1}{4} (2 \sin \alpha \cdot \cos \alpha)^2 \\
 &= \frac{1}{4} \sin^2 2\alpha \\
 &= \frac{1}{4} \left(\frac{1 - \cos 4\alpha}{2} \right) = \frac{1}{8} (1 - \cos 4\alpha)
 \end{aligned}$$

$$\begin{aligned}
 \bullet \sin^3 \alpha \cdot \cos^2 \alpha &= \sin \alpha \cdot (\sin^2 \alpha \cdot \cos^2 \alpha) \\
 &= \sin \alpha \cdot \frac{1}{4} \sin^2 2\alpha \\
 &= \frac{1}{4} \sin \alpha \cdot \frac{1}{2} (1 - \cos 4\alpha) \\
 &= \frac{1}{8} \sin \alpha (1 - \cos 4\alpha) \\
 &= \frac{1}{8} (\sin \alpha - \sin \alpha \cdot \cos 4\alpha) \\
 &= \frac{1}{16} (2 \sin \alpha - 2 \sin \alpha \cdot \cos 4\alpha) \\
 &= \frac{1}{16} (2 \sin \alpha - \sin 5\alpha + \sin 3\alpha)
 \end{aligned}$$

eg:-

$$\begin{aligned}
 \bullet I &= \int \sin^3 \alpha \, d\alpha = \int \sin^2 \alpha \cdot \sin \alpha \, d\alpha \\
 &= \int (1 - \cos^2 \alpha) \sin \alpha \, d\alpha
 \end{aligned}$$

$$\text{Put } t = \cos \alpha \Rightarrow \frac{dt}{d\alpha} = -\sin \alpha \Rightarrow -dt = \sin \alpha \, d\alpha$$

$$\therefore I = \int (1 - t^2) (-dt)$$

$$= -\int (1 - t^2) dt = -t + \frac{t^3}{3} + K$$

$$\Rightarrow I = -\cos \alpha + \frac{1}{3} \cos^3 \alpha + K$$

$$\bullet I = \int \cos^5 \alpha \, d\alpha = \int \cos^4 \alpha \cdot \cos \alpha \, d\alpha$$

$$\Rightarrow I = \int (1 - \sin^2 \alpha)^2 \cos \alpha \, d\alpha$$

$$\text{Put } t = \sin \alpha \Rightarrow \frac{dt}{d\alpha} = \cos \alpha \Rightarrow dt = \cos \alpha \, d\alpha$$

$$\therefore I = \int (1 - t^2)^2 dt$$

$$= \int (1 - 2t^2 + t^4) dt$$

$$= t - \frac{2}{3} t^3 + \frac{1}{5} t^5 + k$$

$$\Rightarrow I = \sin \alpha - \frac{2}{3} \sin^3 \alpha + \frac{1}{5} \sin^5 \alpha + k$$

$$\bullet I = \int \sin^4 \alpha \cdot \cos^3 \alpha \, d\alpha$$

$$= \int \sin^4 \alpha \cdot \cos^2 \alpha \cdot \cos \alpha \, d\alpha$$

$$= \int \sin^4 \alpha \cdot (1 - \sin^2 \alpha) \cos \alpha \, d\alpha$$

$$\text{Put } t = \sin \alpha \Rightarrow \frac{dt}{d\alpha} = \cos \alpha \Rightarrow dt = \cos \alpha \, d\alpha$$

$$\therefore I = \int t^4 (1 - t^2) \, dt$$

$$= \int (t^4 - t^6) \, dt$$

$$= \frac{t^5}{5} - \frac{t^7}{7} + k$$

$$\Rightarrow I = \frac{1}{5} \sin^5 \alpha - \frac{1}{7} \sin^7 \alpha + k$$

$$\bullet I = \int \sin^3 \alpha \cdot \cos^5 \alpha \, d\alpha$$

$$= \int \cos^5 \alpha \cdot \sin^2 \alpha \cdot \sin \alpha \, d\alpha$$

$$= \int \cos^5 \alpha (1 - \cos^2 \alpha) \sin \alpha \, d\alpha$$

$$\text{Put } t = \cos \alpha \Rightarrow \frac{dt}{d\alpha} = -\sin \alpha \Rightarrow -dt = \sin \alpha \cdot d\alpha$$

$$\therefore I = \int t^5 (1 - t^2) \cdot -dt$$

$$= -\int (t^5 - t^7) \, dt = -\left(\frac{t^6}{6} - \frac{t^8}{8}\right) + k$$

$$\Rightarrow I = \frac{1}{8} \cos^8 \alpha - \frac{1}{6} \cos^6 \alpha + k$$

$$\bullet I = \int \frac{\cos^3 \alpha}{\sin^4 \alpha} \, d\alpha = \int \frac{\cos^2 \alpha}{\sin^4 \alpha} \cdot \cos \alpha \, d\alpha$$

$$= \int \frac{1 - \sin^2 \alpha}{\sin^4 \alpha} \cos \alpha \, d\alpha$$

$$\text{Put } t = \sin \alpha \Rightarrow \frac{dt}{d\alpha} = \cos \alpha \Rightarrow dt = \cos \alpha \cdot d\alpha$$

$$\therefore I = \int \left(\frac{1 - t^2}{t^4}\right) dt = \int (t^{-4} - t^{-2}) \, dt$$

$$\frac{t^{-3}}{-3} - \frac{t^{-1}}{-1} + K = -\frac{1}{3}t^{-3} + t^{-1} + K$$

$$\Rightarrow I = -\frac{1}{3} \sin^{-3} x = \frac{1}{3t^3} + \frac{1}{t} + K$$

$$\therefore I = -\frac{1}{3 \sin^3 x} + \frac{1}{\sin x} + K = \operatorname{cosec} x - \frac{1}{3} \operatorname{cosec}^3 x + K$$

$$\bullet I = \int \tan^6 \theta \, d\theta$$

$$= \int \tan^4 \theta \cdot \tan^2 \theta \, d\theta$$

$$= \int \tan^4 \theta (\sec^2 \theta - 1) \, d\theta$$

$$= \int (\tan^4 \theta \cdot \sec^2 \theta - \tan^4 \theta) \, d\theta$$

$$= \int [\tan^4 \theta \cdot \sec^2 \theta - \tan^2 \theta \cdot (\sec^2 \theta - 1)] \, d\theta$$

$$= \int (\tan^4 \theta \cdot \sec^2 \theta - \tan^2 \theta \cdot \sec^2 \theta + \tan^2 \theta) \, d\theta$$

$$= \int (\tan^4 \theta \cdot \sec^2 \theta - \tan^2 \theta \cdot \sec^2 \theta + \sec^2 \theta - 1) \, d\theta$$

$$= \int (\tan^4 \theta \cdot d(\tan \theta) - \int \tan^2 \theta \cdot d(\tan \theta) + \int (\sec^2 \theta - 1) \, d\theta)$$

$$= \frac{1}{5} \tan^5 \theta - \frac{1}{3} \tan^3 \theta + \tan \theta - \theta + K$$

$$\bullet I = \int \frac{\sin 6x + \sin 4x}{\cos 6x + \cos 4x} \, dx$$

$$= \int \frac{2 \sin 5x \cdot \cos x}{2 \cos 5x \cdot \cos x} \, dx$$

$$= \int \tan 5x \, dx$$

$$= \frac{1}{5} \ln |\sec 5x| + K$$

Integration by Trigonometric substitution:-

$$* I = \int \frac{dx}{\sqrt{a^2 - x^2}}$$

$$\text{Put } x = a \sin \theta \Rightarrow \frac{dx}{d\theta} = a \cos \theta \Rightarrow dx = a \cos \theta d\theta$$

$$\begin{aligned} \therefore I &= \int \frac{a \cos \theta d\theta}{\sqrt{a^2 - a^2 \sin^2 \theta}} \\ &= \int \frac{a \cos \theta d\theta}{a \cos \theta} = \int d\theta = \theta + k \end{aligned}$$

$$\Rightarrow I = \sin^{-1}\left(\frac{x}{a}\right) + k$$

$$\Rightarrow \boxed{\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + k}$$

$$* I = \int \frac{dx}{x^2 + a^2}$$

$$\text{Put } x = a \tan \theta \Rightarrow \theta = \tan^{-1}(x/a)$$

$$\Rightarrow \frac{dx}{d\theta} = a \sec^2 \theta \Rightarrow dx = a \sec^2 \theta d\theta$$

$$\therefore I = \int \frac{a \sec^2 \theta d\theta}{a^2 \tan^2 \theta + a^2}$$

$$= \int \frac{a \sec^2 \theta d\theta}{a^2 (\tan^2 \theta + 1)}$$

$$= \frac{1}{a} \int \frac{\sec^2 \theta}{\sec^2 \theta} d\theta = \frac{1}{a} \int d\theta = \frac{1}{a} \theta + k$$

$$\Rightarrow \boxed{\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + k}$$

$$* I = \int \frac{dx}{\sqrt{x^2 + a^2}}$$

$$\text{Put } x = a \tan \theta \Rightarrow \theta = \tan^{-1}(x/a)$$

$$\Rightarrow \frac{dx}{d\theta} = a \sec^2 \theta \Rightarrow dx = a \sec^2 \theta d\theta$$

$$\therefore I = \int \frac{a \sec^2 \theta d\theta}{\sqrt{a^2 \tan^2 \theta + a^2}}$$

$$= \int \frac{a \sec^2 \theta d\theta}{a \sqrt{\sec^2 \theta}}$$

$$= \int \sec \theta d\theta$$

$$= \ln |\sec \theta + \tan \theta| + k$$

Now $\tan \theta = x/a$ and $\sec \theta = \sqrt{\frac{x^2}{a^2} + 1}$

$$\therefore I = \ln \left| \sqrt{\frac{x^2}{a^2} + 1} + \frac{x}{a} \right| + k$$

$$= \ln \left| \frac{\sqrt{x^2 + a^2} + x}{a} \right| + k$$

$$= \ln |\alpha + \sqrt{\alpha^2 + a^2}| + k - \ln a$$

$$= \ln |\alpha + \sqrt{\alpha^2 + a^2}| + c, \quad c = k - \ln a$$

$$\Rightarrow \boxed{\int \frac{dx}{\sqrt{x^2 + a^2}} = \ln |\alpha + \sqrt{\alpha^2 + a^2}| + c}$$

$$* I = \int \frac{dx}{\sqrt{x^2 - a^2}}$$

Put $x = a \sec \theta \Rightarrow \sec \theta = x/a \Rightarrow \tan \theta = \sqrt{\frac{x^2}{a^2} - 1}$

$$\Rightarrow \frac{dx}{d\theta} = a \sec \theta \cdot \tan \theta$$

$$\Rightarrow dx = a \sec \theta \cdot \tan \theta d\theta$$

$$\therefore I = \int \frac{a \sec \theta \cdot \tan \theta d\theta}{\sqrt{a^2 \sec^2 \theta - a^2}}$$

$$= \int \frac{a \sec \theta \cdot \tan \theta d\theta}{a \sqrt{\tan^2 \theta}}$$

$$= \int \sec \theta d\theta$$

$$= \ln |\sec \theta + \tan \theta| + k$$

$$= \ln \left| \frac{x}{a} + \sqrt{\frac{x^2}{a^2} - 1} \right| + k$$

$$= \ln \left| \frac{x + \sqrt{x^2 - a^2}}{a} \right| + k$$

$$= \ln |x + \sqrt{x^2 - a^2}| + k - \ln a$$

$$\therefore \int \frac{dx}{\sqrt{x^2 - a^2}} = \ln |x + \sqrt{x^2 - a^2}| + c, \quad c = k - \ln a$$

$$* I = \int \frac{dx}{x \sqrt{x^2 - a^2}}$$

$$\text{Put } x = a \sec \theta \Rightarrow \theta = \sec^{-1} (x/a)$$

$$\Rightarrow \frac{dx}{d\theta} = a \sec \theta \cdot \tan \theta$$

$$\Rightarrow dx = a \sec \theta \cdot \tan \theta \cdot d\theta$$

$$\therefore I = \int \frac{a \sec \theta \cdot \tan \theta \cdot d\theta}{a \sec \theta \sqrt{a^2 \sec^2 \theta - a^2}}$$

$$= \int \frac{\tan \theta \cdot d\theta}{a \sqrt{\tan^2 \theta}}$$

$$= \int \frac{1}{a} d\theta = \frac{1}{a} \theta + k$$

$$\Rightarrow \int \frac{dx}{x \sqrt{x^2 - a^2}} = \frac{1}{a} \sec^{-1} \left(\frac{x}{a} \right) + k$$

$$\text{e.g.} \cdot I = \int \frac{dx}{\sqrt{25 - 16x^2}}$$

$$= \int \frac{dx}{\sqrt{25 - (4x)^2}}$$

$$= \frac{1}{4} \int \frac{dx}{\sqrt{\left(\frac{5}{4}\right)^2 - x^2}}$$

$$= \frac{1}{4} \sin^{-1} \left(\frac{x}{5/4} \right) + k$$

$$= \frac{1}{4} \sin^{-1} \left(\frac{4x}{5} \right) + k$$

$$\cdot I = \int \frac{e^x}{e^{2x} + 9} dx = \int \frac{e^x dx}{(e^x)^2 + 3^2}$$

$$\text{Put } z = e^x \Rightarrow \frac{dz}{dx} = e^x \Rightarrow dz = e^x dx$$

$$\therefore I = \int \frac{dz}{z^2 + 3^2} = \frac{1}{3} \tan^{-1}(z/3) + C = \frac{1}{3} \tan^{-1}\left(\frac{e^x}{3}\right) + C$$

$$\cdot I = \int \frac{dx}{x\sqrt{x^8 - 4}} = \int \frac{dx}{x\sqrt{(x^4)^2 - 2^2}}$$

$$\text{Put } z = x^4 \Rightarrow \frac{dz}{dx} = 4x^3 \Rightarrow dz = 4x^3 dx$$

$$\therefore I = \int \frac{4x^3 dx}{4x^4 \sqrt{(x^4)^2 - 2^2}}$$

$$= \frac{1}{4} \int \frac{dz}{z \sqrt{z^2 - 2^2}} = \frac{1}{4} \cdot \frac{1}{2} \sec^{-1}\left(\frac{z}{2}\right) + K$$

$$\Rightarrow I = \frac{1}{8} \sec^{-1}\left(\frac{x^4}{2}\right) + K$$

$$\cdot I = \int \frac{\cos \theta d\theta}{\sqrt{4\sin^2 \theta + 1}} = \int \frac{\cos \theta d\theta}{\sqrt{(2\sin \theta)^2 + 1^2}}$$

$$\text{Put } z = 2\sin \theta \Rightarrow \frac{dz}{d\theta} = 2\cos \theta \Rightarrow \frac{dz}{2} = \cos \theta d\theta$$

$$\therefore I = \int \frac{dz/2}{\sqrt{z^2 + 1}} = \frac{1}{2} \int \frac{dz}{\sqrt{z^2 + 1}} = \frac{1}{2} \ln(z + \sqrt{z^2 + 1}) + K$$

$$\Rightarrow I = \frac{1}{2} \ln |2\sin \theta + \sqrt{4\sin^2 \theta + 1}| + K$$

$$= \frac{1}{2} \ln \left| \sin \theta + \sqrt{\sin^2 \theta + \frac{1}{4}} \right| + C; \quad C = K + \ln 2$$

$$\cdot I = \int \frac{x+5}{\sqrt{x^2+6x-7}} dx$$

$$= \int \frac{(x+3)+2}{\sqrt{x^2+6x+9-16}} dx = \int \frac{(x+3)+2}{\sqrt{(x+3)^2 - 4^2}} dx$$

Put $z = \alpha + 3 \Rightarrow \frac{dz}{d\alpha} = 1 \Rightarrow dz = d\alpha$

$$\therefore I = \int \frac{z+2}{\sqrt{z^2-4^2}} dz$$

$$= \int \frac{z dz}{\sqrt{z^2-4^2}} + 2 \int \frac{dz}{\sqrt{z^2-4^2}}$$

$$= \int \frac{1}{2} \frac{d(z^2-4^2)}{\sqrt{z^2-4^2}} + 2 \ln |z + \sqrt{z^2-16}| + k$$

$$= \frac{1}{2} \cdot 2\sqrt{z^2-4^2} + 2 \ln |z + \sqrt{z^2-16}| + k$$

$$= \sqrt{z^2-16} + 2 \ln |z + \sqrt{z^2-16}| + k$$

$$= \sqrt{(\alpha+3)^2-16} + 2 \ln |(\alpha+3) + \sqrt{(\alpha+3)^2-16}| + k$$

$$= \sqrt{\alpha^2+6\alpha-7} + 2 \ln |(\alpha+3) + \sqrt{\alpha^2+6\alpha-7}| + k$$

$$\bullet I = \int \frac{d\alpha}{\sqrt{\alpha} \sqrt{\alpha-a^2}}$$

$$= \int \frac{d\alpha}{\sqrt{\alpha} \sqrt{(\sqrt{\alpha})^2-a^2}}$$

Put $z = \sqrt{\alpha} \Rightarrow \frac{dz}{d\alpha} = \frac{1}{2\sqrt{\alpha}} \Rightarrow 2dz = \frac{d\alpha}{\sqrt{\alpha}}$

$$\therefore I = \int \frac{2dz}{\sqrt{z^2-a^2}}$$

$$= 2 \ln |z + \sqrt{z^2-a^2}| + k$$

$$= 2 \ln |\sqrt{\alpha} + \sqrt{(\sqrt{\alpha})^2-a^2}| + k$$

$$= 2 \ln |\sqrt{\alpha} + \sqrt{\alpha-a^2}| + k$$

$$rb \frac{z+(z+a)}{z^2-(z+a)^2} = rb \frac{z+(z+a)}{z^2-(z+a)^2}$$

Integration By Parts :-

If u and v are two differentiable functions of x , then

$$\frac{d}{dx}(uv) = v \frac{du}{dx} + u \frac{dv}{dx}$$

$$\Rightarrow v \frac{du}{dx} = \frac{d}{dx}(uv) - u \frac{dv}{dx}$$

Integrating both sides -

$$\int v \frac{du}{dx} dx = \int \left[\frac{d}{dx}(uv) - u \frac{dv}{dx} \right] dx$$

$$= \int \frac{d}{dx}(uv) dx - \int u \frac{dv}{dx} dx$$

Let $w = \frac{du}{dx} \Rightarrow u = \int w dx$

Now, $\int v \frac{du}{dx} dx = uv - \int u \frac{dv}{dx} dx$

$$\Rightarrow \int v \cdot w dx = v \int w dx - \int \left[\left(\int w dx \right) \frac{dv}{dx} \right] dx$$

Thus, Integration of Product of two functions

$$= (\text{Int. of 1st}) \times \text{2nd fun} - \text{Int. of (derivation of 2nd} \times \text{Integration of 1st)}$$

i.e. Int. of Product =

$$\text{Second (Int. first)} - \text{Int. of (der. of second} \times \text{Int. of 1st)}$$

The choice of 1st and 2nd funⁿ is based on the rule -

1 - one

E - Exponential Function

T - Trigonometric "

A - Algebraic "

L - Logarithmic "

I - Inverse Trigonometric "

Hence,

$$\int u v dx = v \int u dx - \int \left[\frac{dv}{dx} \int u dx \right] dx$$

This method is known as integration by parts.

u - 1st function and v - 2nd function

And the order of choosing the function -

1 - one

E - Exponential Functions (e^x, a^x)

T - Trigonometric Functions ($\sin x, \cos x$)

A - Algebraic Functions ($x^3, \sqrt{x}, x^{-3/2}$)

L - Logarithmic Functions ($\ln x, \log_e x$)

I - Inverse trigonometric Functions
($\sin^{-1} x, \cos^{-1} x$)

e.g. $I = \int x \cos x dx$

Here x - 2nd Funⁿ, $\cos x$ - 1st Funⁿ

$$\therefore I = x \int \cos x dx - \int \left[\frac{d}{dx} x \int \cos x dx \right] dx$$

$$= x \sin x - \int 1 \cdot \sin x dx$$

$$= x \sin x + \cos x + c$$

• $I = \int x^2 e^x dx$

x^2 - 2nd Funⁿ e^x - 1st Funⁿ

$$\therefore I = x^2 \int e^x dx - \int \left[\frac{d}{dx} x^2 \cdot \int e^x dx \right] dx$$

$$= x^2 e^x - \int 2x \cdot e^x dx$$

$$= x^2 e^x - 2 \int x e^x dx$$

$$= x^2 e^x - 2 \left[x \int e^x dx - \int \left[\frac{d}{dx} x \int e^x dx \right] dx \right]$$

$$\begin{aligned}
 &= \alpha^2 e^\alpha - 2 \left[\alpha e^\alpha - \int e^\alpha dx \right] \\
 &= \alpha^2 e^\alpha - 2 (\alpha e^\alpha - e^\alpha) + C \\
 &= (\alpha^2 - 2\alpha + 2) e^\alpha + C
 \end{aligned}$$

• $I = \int \tan^{-1} x \, dx$

Since $\tan^{-1} x$ is non-integrable, we take 1 as another function.

i.e. $I = \int 1 \cdot \tan^{-1} x \, dx$

Here 1 - 1st funⁿ and $\tan^{-1} x$ = 2nd funⁿ

$$\begin{aligned}
 \therefore I &= (\tan^{-1} x) \int 1 \, dx - \int \left[\frac{d}{dx} (\tan^{-1} x) \int 1 \, dx \right] dx \\
 &= \alpha \tan^{-1} \alpha - \int \frac{1}{1+x^2} \cdot \alpha \, dx \\
 &= \alpha \tan^{-1} \alpha - \frac{1}{2} \int \frac{2x \, dx}{1+x^2} \\
 &= \alpha \tan^{-1} \alpha - \frac{1}{2} \int \frac{d(1+x^2)}{1+x^2} \\
 &= \alpha \tan^{-1} \alpha - \frac{1}{2} \ln(1+x^2) + K
 \end{aligned}$$

• $I = \int (\ln x)^2 \, dx = \int 1 \cdot (\ln x)^2 \, dx$

$$\begin{aligned}
 &= (\ln x)^2 \int 1 \, dx - \int \left[\frac{d}{dx} (\ln x)^2 \int 1 \, dx \right] dx \\
 &= \alpha (\ln \alpha)^2 - \int 2 \ln x \cdot \frac{1}{x} \cdot \alpha \, dx \\
 &= \alpha (\ln \alpha)^2 - 2 \int \ln x \, dx \\
 &= \alpha (\ln \alpha)^2 - 2 \int 1 \cdot \ln x \, dx \\
 &= \alpha (\ln \alpha)^2 - 2 \left[\ln x \int 1 \, dx - \int \left[\frac{d}{dx} (\ln x) \int 1 \, dx \right] dx \right] \\
 &= \alpha (\ln \alpha)^2 - 2 \left(\alpha \ln \alpha - \int \frac{1}{x} \cdot \alpha \, dx \right) \\
 &= \alpha (\ln \alpha)^2 - 2 \left(\alpha \ln \alpha - \int dx \right) \\
 &= \alpha (\ln \alpha)^2 - 2(\alpha \ln \alpha - \alpha) \\
 &= \alpha \left[(\ln \alpha)^2 - 2 \ln \alpha + 2 \right] + K
 \end{aligned}$$

$$* \int e^{\alpha x} f(\alpha) dx = \int e^{\alpha x} dx (f(\alpha)) - \int \left[\frac{d}{dx} f(\alpha) \right] \int e^{\alpha x} dx dx$$

$$= e^{\alpha x} f(\alpha) - \int e^{\alpha x} f'(\alpha) dx + C$$

$$\Rightarrow e^{\alpha x} f(\alpha) = \int e^{\alpha x} f(\alpha) dx + \int e^{\alpha x} f'(\alpha) dx + C$$

$$\Rightarrow e^{\alpha x} f(\alpha) = \int e^{\alpha x} [f(\alpha) + f'(\alpha)] dx + C$$

$$\Rightarrow \boxed{\int e^{\alpha x} (f(\alpha) + f'(\alpha)) dx = e^{\alpha x} f(\alpha) + C}$$

eg: $\int e^{\alpha x} \frac{1 + \sin x}{1 + \cos x} dx$

$$= \int e^{\alpha x} \frac{\sin x}{1 + \cos x} dx + \int \frac{e^{\alpha x}}{1 + \cos x} dx$$

$$= \left(\int e^{\alpha x} dx \right) \left(\frac{\sin x}{1 + \cos x} \right) - \int \left[\frac{d}{dx} \left(\frac{\sin x}{1 + \cos x} \right) \int e^{\alpha x} dx \right] dx$$

$$= e^{\alpha x} \frac{\sin x}{1 + \cos x} - \int \frac{(1 + \cos x) \cos x - \sin x (-\sin x)}{(1 + \cos x)^2} e^{\alpha x} dx + \int \frac{e^{\alpha x}}{1 + \cos x} dx$$

$$+ \int \frac{e^{\alpha x}}{1 + \cos x} dx$$

$$= e^{\alpha x} \frac{\sin x}{1 + \cos x} - \int \frac{\cos x + \cos^2 x + \sin^2 x}{(1 + \cos x)^2} e^{\alpha x} dx + \int \frac{e^{\alpha x}}{1 + \cos x} dx$$

$$= e^{\alpha x} \left(\frac{\sin x}{1 + \cos x} \right) - \int \frac{1 + \cos x}{(1 + \cos x)^2} e^{\alpha x} dx + \int \frac{e^{\alpha x}}{1 + \cos x} dx + C$$

$$= e^{\alpha x} \left(\frac{\sin x}{1 + \cos x} \right) - \int \frac{e^{\alpha x} dx}{1 + \cos x} + \int \frac{e^{\alpha x}}{1 + \cos x} dx + C$$

$$= e^{\alpha x} \left(\frac{\sin x}{1 + \cos x} \right) + C$$

$$* I = \int e^{ax} \cos bx \, dx$$

$$= (\cos bx) \int e^{ax} \, dx - \int \left[\frac{d}{dx} (\cos bx) \int e^{ax} \, dx \right] dx$$

$$= \frac{1}{a} e^{ax} \cos bx - \int (-b \sin bx) \left(\frac{e^{ax}}{a} \right) dx$$

$$= \frac{1}{a} e^{ax} \cos bx + \frac{b}{a} \int e^{ax} \sin bx \, dx$$

$$= \frac{1}{a} e^{ax} \cos bx + \frac{b}{a} \left[\sin bx \int e^{ax} \, dx - \int \left[\frac{d}{dx} (\sin bx) \int e^{ax} \, dx \right] dx \right]$$

$$= \frac{e^{ax}}{a} \cos bx + \frac{b}{a} \left[\frac{1}{a} e^{ax} \sin bx - \int b \cos bx \cdot \frac{1}{a} e^{ax} dx \right]$$

$$= \frac{e^{ax}}{a} \cos bx + \frac{b}{a^2} e^{ax} \sin bx - \frac{b^2}{a^2} \int e^{ax} \cos bx \, dx$$

$$\Rightarrow I = \frac{e^{ax}}{a} \cos bx + \frac{b}{a^2} e^{ax} \sin bx - \frac{b^2}{a^2} I + C$$

$$\Rightarrow I + \frac{b^2}{a^2} I = \left(\frac{a^2 + b^2}{a^2} \right) I = \frac{e^{ax}}{a^2} (a \cos bx + b \sin bx) + C$$

$$\Rightarrow I = \left(\frac{e^{ax}}{a^2 + b^2} \right) (a \cos bx + b \sin bx) + K, \quad K = \frac{Ca^2}{a^2 + b^2}$$

$$\Rightarrow \boxed{\int e^{ax} \cos bx \, dx = \left(\frac{e^{ax}}{a^2 + b^2} \right) (a \cos bx + b \sin bx) + K}$$

similarly,

$$\boxed{\int e^{ax} \sin bx \, dx = \left(\frac{e^{ax}}{a^2 + b^2} \right) (a \sin bx - b \cos bx) + K}$$

$$* I = \int \sqrt{a^2 - x^2} \, dx = \int 1 \cdot \sqrt{a^2 - x^2} \, dx$$

$$= \sqrt{a^2 - x^2} \int 1 \, dx - \int \left[\frac{d}{dx} \sqrt{a^2 - x^2} \cdot \int 1 \, dx \right] dx$$

$$= x \sqrt{a^2 - x^2} - \int \frac{-2x}{2\sqrt{a^2 - x^2}} \cdot x \, dx$$

$$= x \sqrt{a^2 - x^2} + \int \frac{x^2}{\sqrt{a^2 - x^2}} \, dx$$

$$= \alpha \sqrt{a^2 - x^2} + \int \frac{a^2 - (a^2 - x^2)}{\sqrt{a^2 - x^2}} dx$$

$$= \alpha \sqrt{a^2 - x^2} + a^2 \int \frac{dx}{\sqrt{a^2 - x^2}} - \int \sqrt{a^2 - x^2} dx$$

$$= \alpha \sqrt{a^2 - x^2} + a^2 \int \frac{dx}{\sqrt{a^2 - x^2}} - I$$

$$\Rightarrow 2I = 2 \int \sqrt{a^2 - x^2} dx = \alpha \sqrt{a^2 - x^2} + a^2 \sin^{-1}\left(\frac{x}{a}\right) + K$$

$$\Rightarrow \boxed{I = \int \sqrt{a^2 - x^2} dx = \frac{\alpha}{2} \sqrt{a^2 - x^2} + \left(\frac{a^2}{2}\right) \sin^{-1}\left(\frac{x}{a}\right) + K}$$

Similarly,

$$*\int \sqrt{a^2 + x^2} dx = \frac{\alpha}{2} \sqrt{a^2 + x^2} + \left(\frac{a^2}{2}\right) \ln|\alpha + \sqrt{a^2 + x^2}| + K$$

$$*\int \sqrt{x^2 - a^2} dx = \frac{\alpha}{2} \sqrt{x^2 - a^2} - \left(\frac{a^2}{2}\right) \ln|\alpha + \sqrt{x^2 - a^2}| + K$$

eg:-

$$\bullet I = \int x \sin 3x dx$$

$$= \alpha \int \sin 3x dx - \int \left[\frac{d}{dx} \alpha \int \sin 3x dx \right] dx$$

$$= \alpha \int \sin 3x dx - \int \left[\int \sin 3x dx \right] dx$$

$$= \alpha \cdot \frac{1}{4} (3 \sin x - \sin 3x) dx - \int \left[\frac{1}{4} (3 \sin x - \sin 3x) dx \right]$$

$$= \frac{\alpha}{4} \left[3(-\cos x) + \frac{1}{3} \cos 3x \right] - \frac{1}{4} \int (-3 \cos x + \frac{1}{3} \cos 3x) dx$$

$$= \frac{\alpha}{12} (\cos 3x - 9 \cos x) + \frac{1}{4} \int (3 \cos x - \frac{1}{3} \cos 3x) dx$$

$$= \frac{\alpha}{12} (\cos 3x - 9 \cos x) + \frac{1}{4} \left(3 \sin x - \frac{1}{9} \sin 3x \right) + K$$

$$= \frac{\alpha}{12} (\cos 3x - 9 \cos x) + \frac{1}{36} (27 \sin x - \sin 3x) + K$$

$$= \frac{\alpha}{12} \cos 3x - \frac{3}{4} \alpha \cos x + \frac{3}{4} \sin x - \frac{1}{36} \sin 3x + K$$

$$= \frac{1}{4} \left(3 \sin x - 3 \alpha \cos x + \frac{1}{3} \alpha \cos 3x - \frac{1}{9} \sin 3x \right) + K$$

$$\begin{aligned}
 * I &= \int \ln(\alpha^2+1) dx = \int 1 \cdot \ln(\alpha^2+1) dx \\
 &= \ln(\alpha^2+1) \int 1 dx - \int \left[\frac{d}{dx} \ln(\alpha^2+1) \int 1 dx \right] dx \\
 &= \alpha \ln(1+\alpha^2) - \int \frac{2\alpha}{\alpha^2+1} \alpha dx \\
 &= \alpha \ln(1+\alpha^2) - \int \frac{2\alpha^2 dx}{\alpha^2+1} \\
 &= \alpha \ln(1+\alpha^2) - 2 \int \frac{\alpha^2+1-1}{\alpha^2+1} dx \\
 &= \alpha \ln(1+\alpha^2) - 2 \int \left(1 - \frac{1}{\alpha^2+1} \right) dx \\
 &= \alpha \ln(1+\alpha^2) - 2(\alpha - \tan^{-1}\alpha) + K \\
 &= \alpha \ln(1+\alpha^2) - 2\alpha + 2 \tan^{-1}\alpha + K
 \end{aligned}$$

$$\begin{aligned}
 * I &= \int \alpha \sin^{-1}\alpha dx \\
 &= (\sin^{-1}\alpha) \int \alpha dx - \int \left[\frac{d}{dx} (\sin^{-1}\alpha) \int \alpha dx \right] dx \\
 &= \frac{\alpha^2}{2} \sin^{-1}\alpha - \int \frac{1}{\sqrt{1-\alpha^2}} \frac{\alpha^2}{2} dx \\
 &= \frac{\alpha^2}{2} \sin^{-1}\alpha - \frac{1}{2} \int \frac{\alpha^2 dx}{\sqrt{1-\alpha^2}} \\
 &= \frac{\alpha^2}{2} \sin^{-1}\alpha + \frac{1}{2} \int \frac{-\alpha^2 dx}{\sqrt{1-\alpha^2}} \\
 &= \frac{\alpha^2}{2} \sin^{-1}\alpha + \frac{1}{2} \int \frac{-1 + (1-\alpha^2)}{\sqrt{1-\alpha^2}} dx \\
 &\Rightarrow \frac{\alpha^2}{2} \sin^{-1}\alpha + \frac{1}{2} \left[\int \frac{-dx}{\sqrt{1-\alpha^2}} + \int \sqrt{1-\alpha^2} dx \right] \\
 &= \frac{\alpha^2}{2} \sin^{-1}\alpha - \frac{1}{2} \int \frac{dx}{\sqrt{1-\alpha^2}} + \frac{1}{2} \int \sqrt{1-\alpha^2} dx \\
 &= \frac{\alpha^2}{2} \sin^{-1}\alpha - \frac{1}{2} \sin^{-1}\alpha + \frac{1}{2} \left[\frac{\alpha}{2} \sqrt{1-\alpha^2} + \frac{1}{2} \sin^{-1}\alpha \right] + K \\
 &= \frac{\alpha^2}{2} \sin^{-1}\alpha - \frac{1}{2} \sin^{-1}\alpha + \frac{\alpha}{4} \sqrt{1-\alpha^2} + \frac{1}{4} \sin^{-1}\alpha + K \\
 &= \left(\frac{\alpha^2}{2} - \frac{1}{4} \right) \sin^{-1}\alpha + \frac{1}{4} \alpha \sqrt{1-\alpha^2} + K
 \end{aligned}$$

$$\begin{aligned}
 * I &= \int e^x \cos^2 x \, dx \\
 &= (\cos^2 x) \int e^x \, dx - \int \left[\frac{d}{dx} (\cos^2 x) \int e^x \, dx \right] dx \\
 &= e^x \cos^2 x - \int 2 \cos x \cdot (-\sin x) \cdot e^x \, dx \\
 &> e^x \cos^2 x + \int 2 \sin x \cdot \cos x \cdot e^x \, dx \\
 &> e^x \cos^2 x + \int \sin 2x \cdot e^x \, dx \\
 &> e^x \cos^2 x + \left[(\sin 2x) \int e^x \, dx - \int \left[\frac{d}{dx} (\sin 2x) \int e^x \, dx \right] dx \right] \\
 &> e^x \cos^2 x + e^x \sin 2x - \int 2 \cos 2x \cdot e^x \, dx \\
 &> e^x \cos^2 x + e^x \sin 2x - 2 \int e^x \cos 2x \, dx
 \end{aligned}$$

$$\begin{aligned}
 \text{Now } \int e^x \sin 2x \, dx &= e^x \sin 2x - 2 \int e^x \cos 2x \, dx \\
 &> e^x \sin 2x - 2 \left[\cos 2x \cdot \int e^x \, dx - \int \left[\frac{d}{dx} \cos 2x \int e^x \, dx \right] dx \right] \\
 &= e^x \sin 2x - 2 \left[e^x \cos 2x - \int -2 \sin 2x \cdot e^x \, dx \right] \\
 &> e^x \sin 2x - 2 e^x \cos 2x - 2 \int e^x \sin 2x \, dx
 \end{aligned}$$

$$\Rightarrow 5 \int e^x \sin 2x \, dx = e^x \sin 2x - 2 e^x \cos 2x$$

$$\begin{aligned}
 \therefore I &= e^x \cos^2 x + \frac{1}{5} (e^x \sin 2x - 2 e^x \cos 2x) + k \\
 &= e^x \left(\frac{1 + \cos 2x}{2} \right) + \frac{1}{5} (e^x \sin 2x - 2 e^x \cos 2x) + k \\
 &= e^x \left(\frac{1}{2} + \frac{\cos 2x}{2} + \frac{1}{5} \sin 2x - \frac{2}{5} \cos 2x \right) + k \\
 &> \frac{e^x}{10} (5 + 5 \cos 2x + 2 \sin 2x - 4 \cos 2x) + k \\
 &> \frac{e^x}{10} (5 + 2 \sin 2x + \cos 2x) + k
 \end{aligned}$$

$$\begin{aligned}
 \bullet \quad I &= \int \sqrt{9-x^2} \, dx \\
 &= \int \sqrt{3^2-x^2} \, dx \\
 &= \frac{x}{2} \sqrt{3^2-x^2} + \frac{3^2}{2} \sin^{-1}\left(\frac{x}{3}\right) + C \\
 &= \frac{x}{2} \sqrt{9-x^2} + \frac{9}{2} \sin^{-1}\left(\frac{x}{3}\right) + C
 \end{aligned}$$

$$\begin{aligned}
 \bullet \quad I &= \int \sqrt{4x^2+12x+13} \, dx \\
 &= \int \sqrt{4x^2+2 \cdot 2x \cdot 3 + 3^2-3^2+13} \, dx \\
 &= \int \sqrt{(2x)^2+2 \cdot 3 \cdot 2x+3^2+4} \, dx \\
 &= \int \sqrt{(2x+3)^2+2^2} \, dx
 \end{aligned}$$

Put $z = 2x+3 \Rightarrow \frac{dz}{dx} = 2 \Rightarrow \frac{dz}{2} = dx$

$$\begin{aligned}
 \therefore I &= \int \sqrt{z^2+2^2} \frac{dz}{2} \\
 &= \frac{1}{2} \int \sqrt{z^2+2^2} \, dz \\
 &= \frac{1}{2} \left[\frac{z}{2} \sqrt{z^2+2^2} + \frac{2^2}{2} \ln |z + \sqrt{z^2+2^2}| \right] + K \\
 &= \frac{z}{4} \sqrt{z^2+4} + \frac{4}{2 \cdot 2} \ln |z + \sqrt{z^2+4}| + K \\
 &= \left(\frac{2x+3}{4} \right) \sqrt{(2x+3)^2+4} + \ln |(2x+3) + \sqrt{(2x+3)^2+4}| + K \\
 &= \left(\frac{2x+3}{4} \right) \sqrt{4x^2+12x+13} + \ln |2x+3 + \sqrt{4x^2+12x+13}| + K
 \end{aligned}$$

$$\begin{aligned}
 \bullet \quad I &= \int \sqrt{3x^2-2} \, dx \\
 &= \int \sqrt{(\sqrt{3}x)^2 - (\sqrt{2})^2} \, dx
 \end{aligned}$$

Put $t = \sqrt{3}x \Rightarrow \frac{dt}{dx} = \sqrt{3} \Rightarrow \frac{dt}{\sqrt{3}} = dx$

$$\begin{aligned}
 \therefore I &= \int \sqrt{t^2 - (\sqrt{2})^2} \frac{dt}{\sqrt{3}} \\
 &= \frac{1}{\sqrt{3}} \int \sqrt{t^2 - (\sqrt{2})^2} \, dt
 \end{aligned}$$

$$= \frac{1}{\sqrt{3}} \left[\left(\frac{t}{2} \right) \sqrt{t^2 - (\sqrt{2})^2} - \frac{(\sqrt{2})^2}{2} \ln \left| t + \sqrt{t^2 - (\sqrt{2})^2} \right| + K \right]$$

$$= \frac{1}{\sqrt{3}} \left[\left(\frac{\sqrt{3}x}{2} \right) \sqrt{(\sqrt{3}x)^2 - 2} - \ln \left| \sqrt{3}x + \sqrt{(\sqrt{3}x)^2 - (\sqrt{2})^2} \right| \right] + K$$

$$= \frac{x}{2} \sqrt{3x^2 - 2} - \frac{1}{\sqrt{3}} \ln \left| \sqrt{3}x + \sqrt{3x^2 - 2} \right| + K$$

$$* I = \int \frac{\alpha e^\alpha}{(1+\alpha)^2} d\alpha$$

$$= \int e^\alpha \cdot \frac{\alpha}{(1+\alpha)^2} d\alpha$$

$$= \frac{\alpha}{(1+\alpha)^2} \int e^\alpha d\alpha - \int \left(\frac{d}{d\alpha} \frac{\alpha}{(1+\alpha)^2} \int e^\alpha d\alpha \right) d\alpha$$

$$= \frac{\alpha e^\alpha}{(1+\alpha)^2} - \int \frac{(1+\alpha)^2 - 2(1+\alpha) \cdot \alpha}{(1+\alpha)^4} e^\alpha d\alpha$$

$$= \frac{e^\alpha \alpha}{(1+\alpha)^2} - \int \frac{1 + \alpha^2 + 2\alpha - 2\alpha - 2\alpha^2}{(1+\alpha)^4} e^\alpha d\alpha$$

$$= \frac{e^\alpha \alpha}{(1+\alpha)^2} - \int \frac{1 - \alpha^2}{(1+\alpha)^4} e^\alpha d\alpha$$

$$* I = \int \frac{\alpha e^\alpha}{(1+\alpha)^2} d\alpha = \int e^\alpha \frac{\alpha}{(1+\alpha)^2} d\alpha$$

$$= \int e^\alpha \left\{ \frac{1}{1+\alpha} - \frac{1}{(1+\alpha)^2} \right\} d\alpha$$

$$= \int \left(\frac{e^\alpha}{1+\alpha} - \frac{e^\alpha}{(1+\alpha)^2} \right) d\alpha$$

$$= \left(\frac{1}{1+\alpha} \right) \int e^\alpha d\alpha - \int \left[\frac{d}{d\alpha} \left(\frac{1}{1+\alpha} \right) \int e^\alpha d\alpha \right] d\alpha - \int \frac{e^\alpha}{(1+\alpha)^2} d\alpha$$

$$= e^\alpha \left(\frac{1}{1+\alpha} \right) - \int \frac{-1}{(1+\alpha)^2} e^\alpha d\alpha - \int \frac{e^\alpha}{(1+\alpha)^2} d\alpha + K$$

$$= e^\alpha \left(\frac{1}{1+\alpha} \right) + \int \frac{e^\alpha}{(1+\alpha)^2} d\alpha - \int \frac{e^\alpha}{(1+\alpha)^2} d\alpha + K$$

$$= e^\alpha \left(\frac{1}{1+\alpha} \right) + K$$

Partial Fractions And Integration of Rational Functions :-

A function of the form $\frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials is called a rational function.

• Degree of $P(x) <$ Degree of $Q(x)$

\Rightarrow Proper rational function

• Degree of $P(x) >$ Degree of $Q(x)$

\Rightarrow Improper rational function

$$\text{then } \frac{P(x)}{Q(x)} = \theta(x) + \frac{R(x)}{Q(x)}$$

where $\deg R(x) < \deg Q(x) \Rightarrow$ A proper rational function.

A proper fraction can be decomposed into simpler functions, called partial fractions, and each simpler fraction can be integrated separately by the proper methods.

Now four different cases arises depending on the factors of the denominator $Q(x)$.

* The partial fraction corresponding to every non-repeated linear fraction factor $(ax+b)$ of $Q(x)$ is of the form $\frac{A}{ax+b}$, where A is a constant

If $Q(x) = (a_1x+b_1)(a_2x+b_2)(a_3x+b_3)$, then

$$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \frac{A_3}{a_3x+b_3}$$

* To a repeated linear factor of the form $(ax+b)^n$, the ~~corres~~ corresponding n partial

fractions of the form $\frac{A_n}{(ax+b)^n}$, $n=1, 2, \dots, n$

$$\text{If } Q(x) = (ax+b)^3 (cx+d)$$

$$\frac{R(x)}{Q(x)} = \frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \frac{A_3}{(ax+b)^3} + \frac{A_4}{cx+d}$$

* To a non-repeated quadratic factor (lx^2+px+q) , there corresponding a partial fraction of the form $\frac{Ax+B}{lx^2+px+q}$.

$$\text{If } Q(x) = (lx^2+px+q)(ax+b)$$

$$\frac{R(x)}{Q(x)} = \frac{Ax+B}{lx^2+px+q} + \frac{A_2}{ax+b}$$

* To a repeated quadratic factor $(lx^2+px+q)^n$, there corresponding n partial fractions of the form $\frac{A_n x + B_n}{(lx^2+px+q)^n}$, $n=1, 2, \dots, n$

$$\text{If } Q(x) = (lx^2+px+q)^2 (ax+b)$$

$$\frac{R(x)}{Q(x)} = \frac{Ax+B_1}{lx^2+px+q} + \frac{Ax+B_2}{(lx^2+px+q)^2} + \frac{A_3}{ax+b}$$

$$\text{eg: } \int \frac{4x+5}{x^2+x-2} dx$$

The integral $\frac{4x+5}{x^2+x-2}$ is a rational function.

$$\text{Thus, } \frac{P(x)}{Q(x)} = \frac{4x+5}{x^2+x-2}$$

$$\begin{aligned} \text{Now } Q(x) &= x^2+x-2 = x^2+2x-x-2 \\ &= x(x+2)-1(x+2) \\ &= (x+2)(x-1) \end{aligned}$$

$$\therefore \frac{P(x)}{Q(x)} = \frac{4x+5}{x^2+x-2} = \frac{4x+5}{(x+2)(x-1)}$$

$$\text{Now } \frac{4x+5}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1}$$

$$= \frac{A(x-1) + B(x+2)}{(x+2)(x-1)}$$

$$\Rightarrow 4x+5 = A(x-1) + B(x+2)$$

For finding the constants :-

Method - 1 :-

$$4x+5 = A(x-1) + B(x+2) = Ax - A + Bx + 2B$$

$$\Rightarrow 4x+5 = (A+B)x + (2B-A)$$

on comparing -

$$\begin{cases} A+B=4 \\ 2B-A=5 \end{cases}$$

$$\underline{\hspace{10em}} \quad 3B=9 \Rightarrow \boxed{B=3}$$

$$\therefore A=4-B \Rightarrow \boxed{A=1}$$

Method - 2 :-

$$4x+5 = A(x-1) + B(x+2)$$

Put $x=1 \Rightarrow 9 = 0 + 3B \Rightarrow \boxed{B=3}$

$x=-2 \Rightarrow -3 = -3A + 0 \Rightarrow \boxed{A=1}$

Hence

$$\frac{4x+5}{x^2+x-2} = \frac{1}{x+2} + \frac{3}{x-1}$$

$$\therefore \int \left(\frac{4x+5}{x^2+x-2} \right) dx = \int \left(\frac{1}{x+2} + \frac{3}{x-1} \right) dx$$

$$= \ln(x+2) + 3\ln(x-1) + C$$

$$= \ln |(x+2) \cdot (x-1)^3| + C$$

$$I = \int \frac{x^2}{(x+1)^2(x-2)} dx$$

$$\frac{x^2}{(x+1)^2(x-2)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-2}$$

$$= A(x+1) + B + C(x-2)$$

$$= \frac{A(x+1)(x-2) + B(x-2) + C(x+1)^2}{(x+1)^2(x-2)}$$

$$\Rightarrow x^2 = A(x+1)(x-2) + B(x-2) + C(x+1)^2$$

$$\text{Put } x=2 \Rightarrow 4 = (2+1)^2 C \Rightarrow 4 = 9C \Rightarrow \boxed{C = 4/9}$$

$$x=-1 \Rightarrow 1 = (-1-2)B \Rightarrow \boxed{B = -1/3}$$

$$x=0 \Rightarrow 0 = -2A - 2B + C$$

$$\Rightarrow 2A = C - 2B = \frac{4}{9} + \frac{2}{3} = \frac{10}{9}$$

$$\Rightarrow \boxed{A = 5/9}$$

$$\therefore \frac{x^2}{(x+1)^2(x-2)} = \frac{5/9}{x+1} + \frac{-1/3}{(x+1)^2} + \frac{4/9}{x-2}$$

$$\text{Now, } I = \int \frac{x^2}{(x+1)^2(x-2)} dx = \int \left[\frac{5/9}{x+1} + \frac{-1/3}{(x+1)^2} + \frac{4/9}{x-2} \right] dx$$

$$= \frac{5}{9} \int \frac{dx}{x+1} - \frac{1}{3} \int \frac{dx}{(x+1)^2} + \frac{4}{9} \int \frac{dx}{x-2}$$

$$= \frac{5}{9} \ln|x+1| - \frac{1}{3} \int \frac{d(x+1)}{(x+1)^2} + \frac{4}{9} \ln|x-2| + C$$

$$= \frac{5}{9} \ln|x+1| - \frac{1}{3} \left(\frac{1}{x+1} \right) + \frac{4}{9} \ln|x-2| + C$$

$$* I = \int \frac{2x^2 + x + 3}{(x^2 + 2)(x-1)} dx$$

$$\frac{2x^2 + x + 3}{(x^2 + 2)(x-1)} = \frac{Ax + B}{x^2 + 2} + \frac{C}{x-1} = \frac{(Ax + B)(x-1) + C(x^2 + 2)}{(x^2 + 2)(x-1)}$$

$$\Rightarrow 2x^2 + x + 3 = (Ax + B)(x-1) + C(x^2 + 2)$$

$$= A\alpha^2 - A\alpha + B\alpha - B + C\alpha^2 + 2C$$

$$= (A+C)\alpha^2 + (B-A)\alpha + (2C-B)$$

On comparing -

$$A+C=2, \quad B-A=1 \quad \text{and} \quad 2C-B=3$$

$$(B-A) + (2C-B) = 1+3 \Rightarrow 2C-A=4$$

$$(A+C) + (2C-A) = 2+4 \Rightarrow 3C=6 \Rightarrow \boxed{C=2}$$

$$\therefore A=2-C \Rightarrow A=2-2=0 \Rightarrow \boxed{A=0}$$

$$\text{and } B=1+A \Rightarrow B=0+1 \Rightarrow \boxed{B=1}$$

$$\text{Now } \frac{2\alpha^2 + \alpha + 3}{(\alpha^2+2)(\alpha-1)} = \frac{1}{\alpha^2+2} + \frac{2}{\alpha-1}$$

$$I = \int \frac{2\alpha^2 + \alpha + 3}{(\alpha^2+2)(\alpha-1)} d\alpha = \int \left[\frac{1}{(\alpha^2+2)} + \frac{2}{\alpha-1} \right] d\alpha$$

$$= \int \frac{d\alpha}{\alpha^2+2} + 2 \int \frac{d\alpha}{\alpha-1}$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{\alpha}{\sqrt{2}}\right) + 2 \ln|\alpha-1| + K$$

$$* I = \int \frac{4\alpha^2}{(\alpha-3)(\alpha+1)} d\alpha$$

Here $\frac{4\alpha^2}{(\alpha-3)(\alpha+1)}$ is not a proper fraction, so we

first convert it to a proper fraction -

$$\frac{4\alpha^2}{(\alpha-3)(\alpha+1)} = 4 + \frac{A}{\alpha-3} + \frac{B}{\alpha+1}$$

$$= \frac{4(\alpha-3)(\alpha+1) + A(\alpha+1) + B(\alpha-3)}{(\alpha-3)(\alpha+1)}$$

$$\Rightarrow 4\alpha^2 = 4(\alpha-3)(\alpha+1) + A(\alpha+1) + B(\alpha-3)$$

$$\text{Put } \alpha = -1 \Rightarrow 4 = -4B \Rightarrow \boxed{B = -1}$$

$$\alpha = 3 \Rightarrow 36 = 4A \Rightarrow \boxed{A = 9}$$

$$\therefore \frac{4x^2}{(x-3)(x+1)} = 4 + \frac{9}{x-3} - \frac{1}{x+1}$$

$$\begin{aligned} \Rightarrow I &= \int \frac{4x^2}{(x-3)(x+1)} dx = \int \left(4 + \frac{9}{x-3} - \frac{1}{x+1} \right) dx \\ &= \int 4 dx + 9 \int \frac{dx}{x-3} - \int \frac{dx}{x+1} \\ &= 4x + 9 \ln|x-3| - \ln|x+1| + K \end{aligned}$$

$$* I = \int \frac{dx}{x^2 - a^2} = \int \frac{dx}{(x+a)(x-a)}$$

$$\text{Now } \frac{1}{(x+a)(x-a)} = \frac{A}{x+a} + \frac{B}{x-a} = \frac{A(x-a) + B(x+a)}{(x+a)(x-a)}$$

$$\Rightarrow 1 = (x-a)A + B(x+a)$$

$$\text{Put } x = a \Rightarrow 1 = 2aB \Rightarrow \boxed{B = 1/2a}$$

$$x = -a \Rightarrow 1 = -2aA \Rightarrow \boxed{A = -1/2a}$$

$$\therefore \frac{1}{(x+a)(x-a)} = \frac{-1/2a}{x+a} + \frac{1/2a}{x-a} = \frac{1}{2a} \left(\frac{1}{x-a} - \frac{1}{x+a} \right)$$

$$\Rightarrow I = \int \frac{dx}{(x+a)(x-a)} = \int \frac{1}{2a} \left(\frac{1}{x-a} - \frac{1}{x+a} \right) dx$$

$$= \frac{1}{2a} \left[\int \frac{dx}{x-a} - \int \frac{dx}{x+a} \right]$$

$$= \frac{1}{2a} \left[\ln|x-a| - \ln|x+a| \right] + C$$

$$= \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + C$$

$$\therefore \boxed{\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + C}$$

$$* I = \int \frac{dx}{(x+a)(x+b)}$$

$$\frac{1}{(x+a)(x+b)} = \frac{A}{x+a} + \frac{B}{x+b} \Rightarrow \frac{A(x+b) + B(x+a)}{(x+a)(x+b)}$$

$$\Rightarrow 1 = A(x+b) + B(x+a)$$

$$\text{Put } x = -b \Rightarrow 1 = (-b+a)B \Rightarrow B = \frac{1}{a-b}$$

$$x = -a \Rightarrow 1 = (-a+b)A \Rightarrow A = -\frac{1}{a-b}$$

$$\therefore \frac{1}{(x+a)(x-b)} = \frac{1/a-b}{x+b} - \frac{1/a-b}{x+a} = \left(\frac{1}{a-b}\right) \left(\frac{1}{x+b} - \frac{1}{x+a}\right)$$

$$\Rightarrow I = \int \frac{dx}{(x+a)(x-b)} = \int \frac{1}{a-b} \left[\frac{1}{x+b} - \frac{1}{x+a} \right] dx$$

$$= \left(\frac{1}{a-b}\right) \left[\int \frac{dx}{x+b} - \int \frac{dx}{x+a} \right]$$

$$= \left(\frac{1}{a-b}\right) \left[\ln|x+b| - \ln|x+a| \right] + K$$

$$= \left(\frac{1}{a-b}\right) \ln \left| \frac{x+b}{x+a} \right| + K$$

$$\therefore \boxed{\int \frac{dx}{(x+a)(x-b)} = \frac{1}{a-b} \ln \left| \frac{x+b}{x+a} \right| + K}$$

$$* I = \int \frac{dx}{a^2-x^2} = \int \frac{-dx}{(x^2-a^2)} = - \int \frac{dx}{x^2-a^2}$$

$$= -\frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + C$$

$$= \frac{1}{2a} \ln \left| \frac{x+a}{x-a} \right| + C$$

$$\Rightarrow \boxed{\int \frac{dx}{a^2-x^2} = \frac{1}{2a} \ln \left| \frac{x+a}{x-a} \right| + C}$$

eg. $\cdot I = \int \frac{20x+3}{6x^2-x-2} dx$

$\frac{20x+3}{6x^2-x-2} = \frac{p(x)}{q(x)}$ which is a proper fraction.

Now $6x^2-x-2 = 6x^2-4x+3x-2$
 $= 2x(3x-2)+1(3x-2)$
 $= (3x-2)(2x+1)$

$\Rightarrow 6x^2-x-2 = (3x-2)(2x+1)$

$\therefore \frac{20x+3}{6x^2-x-2} = \frac{20x+3}{(3x-2)(2x+1)}$

$\Rightarrow \frac{20x+3}{(3x-2)(2x+1)} = \frac{A}{3x-2} + \frac{B}{2x+1} = \frac{A(2x+1)+B(3x-2)}{(3x-2)(2x+1)}$

$\Rightarrow 20x+3 = A(2x+1) + B(3x-2)$

Put $x = -\frac{1}{2} \Rightarrow -10+3 = (-\frac{3}{2}-2)B$

$\Rightarrow -7 = -\frac{7}{2}B \Rightarrow \boxed{B=2}$

$x = \frac{2}{3} \Rightarrow \frac{40}{3}+3 = (\frac{4}{3}+1)A$

$\Rightarrow \frac{49}{3} = \frac{7}{3}A \Rightarrow \boxed{A=7}$

$\therefore \frac{20x+3}{(3x-2)(2x+1)} = \frac{7}{3x-2} + \frac{2}{2x+1}$

$\Rightarrow I = \int \frac{20x+3}{6x^2-x-2} dx = \int \left(\frac{7}{3x-2} + \frac{2}{2x+1} \right) dx$

$= 7 \int \frac{dx}{3x-2} + 2 \int \frac{dx}{2x+1}$

$= \frac{7}{3} \ln|3x-2| + \ln|2x+1| + K$

* $I = \int \frac{x^4+3x^3+x^2-1}{x^3+x^2-x-1} dx$

Here the fraction is improper. Hence we have

To first convert it to proper:

$$\frac{x^4 + 3x^3 + x^2 - 1}{x^3 + x^2 - x - 1} = Q(x) + \frac{P(x)}{D(x)}$$

$$\Rightarrow \frac{x^4 + 3x^3 + x^2 - 1}{x^3 + x^2 - x - 1} = \frac{x^4 + 3x^3 + x^2 - 1}{(x+1)^2(x-1)}$$

$$\therefore x^3 + x^2 - x - 1 = x^2(x+1) - 1(x+1)$$

$$= (x+1)(x^2-1)$$

$$= (x+1)(x-1)(x+1)$$

$$= (x+1)^2(x-1)$$

$$\Rightarrow x^3 + x^2 - x - 1 = (x+1)^2(x-1)$$

$$\text{Now } \frac{x^4 + 3x^3 + x^2 - 1}{x^3 + x^2 - x - 1} = (Ax+B) + \frac{A_1}{x+1} + \frac{A_2}{(x+1)^2} + \frac{A_3}{x-1}$$

$$\Rightarrow x^4 + 3x^3 + x^2 - 1 = (Ax+B)(x+1)^2(x-1) + A_1(x+1)(x-1) + A_2(x-1) + A_3(x+1)^2$$

$$\text{Put } x = -1 \Rightarrow -2 = -2A_2 \Rightarrow \boxed{A_2 = 1}$$

$$x = 1 \Rightarrow 4 = 4A_3 \Rightarrow \boxed{A_3 = 1}$$

$$x = 0 \Rightarrow -1 = -B - A_1 - A_2 + A_3$$

$$\Rightarrow -1 = -B - A_1 - 1 + 1 \Rightarrow A_1 + B = 1$$

$$\Rightarrow -1 = -B - A_1 - 1 + 1 \Rightarrow A_1 + B = 1$$

$$\therefore x^4 + 3x^3 + x^2 - 1 = (Ax+B)(x+1)^2(x-1) + (x+1)(x-1)(B+1) + (x-1) + (x+1)^2$$

$$= (Ax+B)(x^2+2x+1)(x-1) + (x^2-1)(B+1) + x-1 + x^2+2x+1$$

$$= (Ax+B)(x^3+2x^2+x-x^2-2x-1) + [x^2B+B+x^2] + x^2+3x$$

$$= (Ax+B)(x^3+x^2-x-1) + x^2(B+1) + 3x(B+1) + B + x^2+3x$$

$$= Ax^4 + Ax^3 - Ax^2 - Ax + Bx^3 + Bx^2 - Bx - B + x^3 + 3x^2 + 3x + B + x^2 + 3x$$

$$= A\alpha^4 + (A+B)\alpha^3 + (2-A)\alpha^2 + (-A-B+3)\alpha - 2B + 1$$

Now by comparing -

$$\boxed{A=1} \text{ and } B+A=3 \rightarrow B=3-A \Rightarrow \boxed{B=2}$$

$$\therefore \cancel{A=B+1=2-1} \Rightarrow \boxed{A_2=1} \quad A_1=1-B \Rightarrow \boxed{A_1=-1}$$

$$\text{Now, } \frac{\alpha^4 + 3\alpha^2 + \alpha^2 - 1}{(\alpha+1)^2(\alpha-1)} = (\alpha+2) + \frac{-1}{\alpha+1} + \frac{1}{(\alpha+1)^2} + \frac{1}{(\alpha-1)}$$

$$\Rightarrow I = \int \frac{(\alpha^4 + 3\alpha^2 + \alpha^2 - 1) d\alpha}{(\alpha+1)^2(\alpha-1)} = \int \left[(\alpha+2) - \frac{1}{\alpha+1} + \frac{1}{(\alpha+1)^2} + \frac{1}{(\alpha-1)} \right] d\alpha$$

$$= \int (\alpha+2) d\alpha - \int \frac{d\alpha}{\alpha+1} + \int \frac{d\alpha}{(\alpha+1)^2} + \int \frac{d\alpha}{(\alpha-1)}$$

$$= \frac{\alpha^2}{2} + 2\alpha - \ln|\alpha+1| + \int \frac{d(\alpha+1)}{(\alpha+1)^2} + \ln|\alpha-1| + C$$

$$= \frac{\alpha^2}{2} + 2\alpha - \ln|\alpha+1| + \ln|\alpha-1| - \frac{1}{\alpha+1} + C$$

$$= \ln|\alpha-1| - \ln|\alpha+1| + 2\alpha + \frac{\alpha^2}{2} - \frac{1}{\alpha+1} + C$$

$$* I = \int \frac{e^x dx}{e^{2x} + 3e^x + 1} = \int \frac{e^x dx}{(e^x)^2 + 3e^x + 1}$$

$$\text{Put } z = e^x \Rightarrow \frac{dz}{dx} = e^x \Rightarrow dz = e^x dx$$

$$\therefore I = \int \frac{dz}{z^2 + 3z + 1} = \int \frac{dz}{z^2 + 2 \cdot z \cdot \frac{3}{2} + (\frac{3}{2})^2 - (\frac{3}{2})^2 + 1}$$

$$\text{Now } z^2 + 3z + 1 = z^2 +$$

$$= \int \frac{dz}{(z + \frac{3}{2})^2 - \frac{9}{4} + 1}$$

$$= \int \frac{dz}{(z + \frac{3}{2})^2 - \frac{5}{4}}$$

$$= \int \frac{dz}{(z + \frac{3}{2})^2 - (\frac{\sqrt{5}}{2})^2}$$

$$= \frac{1}{2(\frac{\sqrt{5}}{2})} \ln \left| \frac{z + \frac{3}{2} - \frac{\sqrt{5}}{2}}{z + \frac{3}{2} + \frac{\sqrt{5}}{2}} \right| + K$$

$$= \frac{1}{\sqrt{5}} \ln \left| \frac{2z + (3 - \sqrt{5})}{2z + (3 + \sqrt{5})} \right| + K$$

$$= \frac{1}{\sqrt{5}} \ln \left| \frac{2e^x + (3 - \sqrt{5})}{2e^x + (3 + \sqrt{5})} \right| + K$$

$$* I = \int \frac{x-5}{7-x^2+6x} dx = - \int \frac{(x-5) dx}{x^2-6x-7}$$

$$= - \int \frac{(x-5) dx}{x^2 + 2 \cdot 3x + 3^2 - 3^2 - 7}$$

$$= - \int \frac{(x-5) dx}{(x-3)^2 - 9 - 7}$$

$$= - \int \frac{(x-5) dx}{(x-3)^2 - 4^2}$$

$$= - \int \frac{[(x-3) - 2] dx}{(x-3)^2 - 4^2}$$

$$= -\int \frac{(x-3) dx}{(x-3)^2 - 4^2} + \int \frac{2 dx}{(x-3)^2 - 4^2}$$

$$= -\frac{1}{2} \int \frac{d[(x-3)-4^2]}{(x-3)^2 - 4^2} + \frac{2}{2 \cdot 4} \ln \left| \frac{(x-3)-4}{(x-3)+4} \right| + C$$

$$= \frac{1}{2} \ln |[(x-3)^2 - 4^2]| + \frac{1}{4} \ln \left| \frac{x-7}{x+1} \right| + C$$

$$= \frac{1}{2} \ln |x^2 - 6x - 7| + \frac{1}{4} \ln \left| \frac{x-7}{x+1} \right| + C$$

$$* I = \int \frac{dx}{\sin x (3+2\cos x)}$$

Put $z = \cos x \Rightarrow \frac{dz}{dx} = -\sin x \Rightarrow dz = -\sin x dx$

$$\therefore I = \int \frac{\sin x dx}{\sin^2 x (3+2\cos x)}$$

$$= \int \frac{\sin x dx}{(1-\cos^2 x)(3+2\cos x)}$$

$$= \int \frac{-dz}{(1-z^2)(3+2z)} = \int \frac{dz}{(z^2-1)(3+2z)} = \int \frac{dz}{(z+1)(z-1)(3+2z)}$$

Now, $\frac{1}{(z+1)(z-1)(3+2z)} = \frac{A}{z+1} + \frac{B}{z-1} + \frac{C}{3+2z}$

$$\Rightarrow 1 = A(z-1)(3+2z) + B(z+1)(3+2z) + C(z+1)(z-1)$$

Put $z = 1 \Rightarrow 1 = 10B \Rightarrow \boxed{B = 1/10}$

$z = -1 \Rightarrow 1 = -2A \Rightarrow \boxed{A = -1/2}$

$z = -3/2 \Rightarrow 1 = C(5/2) \Rightarrow \boxed{C = 4/5}$

$$\therefore \frac{1}{(z+1)(z-1)(3+2z)} = \frac{-1/2}{z+1} + \frac{1/10}{z-1} + \frac{4/5}{3+2z}$$

$$\Rightarrow I = \int \frac{dz}{(z+1)(z-1)(3+2z)} = \frac{1}{2} \int \frac{dz}{z+1} + \frac{1}{10} \int \frac{dz}{z-1} + \frac{4}{5} \int \frac{dz}{3+2z}$$

$$\Rightarrow I = \frac{1}{2} \ln|1+z| + \frac{1}{10} \ln|z-1| + \frac{2}{5} \ln|3+2z| + C$$

$$\Rightarrow I = \frac{1}{2} \ln|1+\cos x| + \frac{1}{10} \ln|\cos x - 1| + \frac{2}{5} \ln|3+2\cos x| + C$$

Definite Integral :-

Let us consider a function $f(x)$ continuous in the interval $[a, b]$. Let the interval be divided into n -subintervals of length h_1, h_2, \dots, h_n by the points $\alpha_1, \alpha_2, \dots, \alpha_{n-1}$ s.t

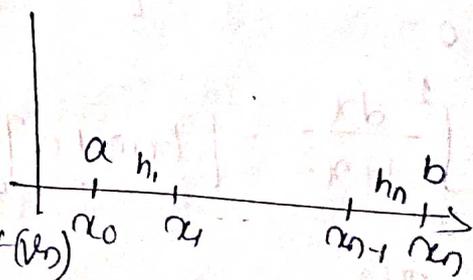
$$a = \alpha_0 < \alpha_1 < \alpha_2 < \dots < \alpha_{n-1} < \alpha_n = b$$

$$\text{Here } h_r = \alpha_r - \alpha_{r-1}, \quad r = 1, 2, \dots, n$$

Let v_r be any point in

$$[\alpha_{r-1}, \alpha_r], \quad r = 1, 2, \dots, n$$

$$\text{Let } S = h_1 f(v_1) + h_2 f(v_2) + \dots + h_n f(v_n)$$



$$\Rightarrow \lim_{n \rightarrow \infty} S = \int_a^b f(x) dx$$

i.e. The limit of S exists as $n \rightarrow \infty$ and $\max. h_r \rightarrow 0$, then the limit is called the definite integral of $f(x)$ from a to b .

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} [h_1 f(v_1) + h_2 f(v_2) + \dots + h_n f(v_n)]$$

where a - lower limit and b - upper limit

Fundamental Theorem of Integral Calculus:

If $f(x)$ is continuous in the interval $[a, b]$ and $F(x)$ is an anti-derivative of $f(x)$, then

$$\int_a^b f(x) dx = F(b) - F(a)$$

$$* \int_a^b [f(x) \pm g(x)] dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx$$

$$* \int_a^b \lambda f(x) dx = \lambda \int_a^b f(x) dx, \quad \lambda = \text{constant}$$

e.g.:- $\int_1^2 x^3 dx = \left[\frac{x^4}{4} \right]_1^2 = \frac{2^4}{4} - \frac{1^4}{4} = \frac{16}{4} - \frac{1}{4} = \frac{15}{4}$

$$\bullet \int_0^{\pi/2} \sin x dx = [-\cos x]_0^{\pi/2} = [-\cos \pi/2 - \cos 0] = -(0 - 1) = 1$$

$$\bullet \int_0^1 \frac{dx}{1+x^2} = [\tan^{-1} x]_0^1 = \tan^{-1} 1 - \tan^{-1} 0 = \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

$$\bullet \int_0^{\pi/2} x \cos x dx = [x \cdot \sin x - \int \sin x dx]_0^{\pi/2}$$
$$= [x \cdot \sin x + \cos x]_0^{\pi/2}$$

$$= \frac{\pi}{2} \sin \frac{\pi}{2} + \cos \frac{\pi}{2} - 0 - \cos 0$$

$$= \frac{\pi}{2} + 0 - 1 = \frac{\pi}{2} - 1$$

$$\bullet \int_2^3 2x e^{x^2} dx = \int_4^9 e^z dz = [e^z]_4^9 = e^9 - e^4$$

$$\text{as } x^2 = z \Rightarrow dz = 2x dx$$

$$\text{and } x \rightarrow 2 \Rightarrow z \rightarrow x^2 = 2^2 = 4$$

$$x \rightarrow 3 \Rightarrow z \rightarrow 3^2 = 9$$

$$* I = \int_0^{\pi/4} \sin^5 x \cos x \, dx$$

$$\text{Put } z = \sin x \Rightarrow dz = \cos x \, dx$$

$$x \rightarrow 0, \pi/4 \Rightarrow z \rightarrow \sin 0, \sin \pi/4 \Rightarrow z \rightarrow 0, \frac{1}{\sqrt{2}}$$

$$\therefore I = \int_0^{1/\sqrt{2}} z^5 \, dz = \left[\frac{z^6}{6} \right]_0^{1/\sqrt{2}} = \frac{1}{6 \cdot 6} - 0 = \frac{1}{48}$$

Properties :-

$$* \int_a^b f(x) \, dx = \int_a^b f(t) \, dt = \int_a^b f(z) \, dz$$

$$* \int_a^b f(x) \, dx = - \int_b^a f(x) \, dx$$

$$* \int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx, \quad a < c < b.$$

Proof :- $* \int_a^b f(x) \, dx = F(b) - F(a)$

$$\int_a^b f(t) \, dt = F(b) - F(a)$$

$$\int_a^b f(z) \, dz = F(b) - F(a)$$

$$\therefore \boxed{\int_a^b f(x) \, dx = \int_a^b f(t) \, dt = \int_a^b f(z) \, dz}$$

$$* \int_a^b f(x) \, dx = F(b) - F(a)$$

$$= - [F(a) - F(b)]$$

$$= - \int_b^a f(x) \, dx$$

$$\Rightarrow \int_a^b f(x) \, dx = - \int_b^a f(x) \, dx$$

$$\int_a^c f(x) dx + \int_c^b f(x) dx = [F(x)]_a^c + [F(x)]_c^b$$

$$= F(c) - F(a) + F(b) - F(c)$$

$$= F(b) - F(a)$$

$$= \int_a^b f(x) dx$$

$$\therefore \int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$$

e.g. • $I = \int_1^4 [x] dx$

$$= \int_1^2 [x] dx + \int_2^3 [x] dx + \int_3^4 [x] dx$$

$$= \int_1^2 1 dx + \int_2^3 2 dx + \int_3^4 3 dx$$

$$= [x]_1^2 + [2x]_2^3 + [3x]_3^4$$

$$= (2-1) + (6-4) + (12-9)$$

$$= 6$$

• $\int_{-3}^4 |x| dx = \int_{-3}^0 |x| dx + \int_0^4 |x| dx$

$$= \int_{-3}^0 -x dx + \int_0^4 x dx$$

$$= \left[-\frac{x^2}{2} \right]_{-3}^0 + \left[\frac{x^2}{2} \right]_0^4$$

$$= -0 + \frac{9}{2} + \frac{16}{2} - 0 = \frac{25}{2}$$

Properties :-

$$* \int_a^b f(x) dx = \int_a^b f(a+b-x) dx$$

$$\text{In Particular, } \int_0^a f(x) dx = \int_0^a f(a-x) dx$$

$$* \int_{-a}^a f(x) dx = \begin{cases} 2 \int_0^a f(x) dx, & f \text{ is even function.} \\ 0, & f \text{ is odd function.} \end{cases}$$

$$* \int_0^{2a} f(x) dx = \begin{cases} 2 \int_0^a f(x) dx, & f(2a-x) = f(x) \\ 0, & f(2a-x) = -f(x) \end{cases}$$

Proof :-

$$* \text{ Put } z = a+b-x, \text{ then } dz = -dx$$

$$x \rightarrow a \Rightarrow z \rightarrow b \text{ and } x \rightarrow b \Rightarrow z \rightarrow a$$

$$\therefore \int_a^b f(a+b-x) dx = \int_b^a f(z) (-dz) = \int_a^b f(z) dz$$

$$\Rightarrow \int_a^b f(a+b-x) dx = \int_a^b f(x) dx$$

$$* \int_{-a}^0 f(x) dx = I$$

$$\text{Put } x = -v \Rightarrow dx = -dv \text{ then } x \rightarrow -a, 0 \Rightarrow v \rightarrow a, 0$$

$$\therefore I = \int_a^0 f(-v) (-dv) = \int_0^a f(-v) dv = \int_0^a f(-x) dx$$

For an even function $f(-x) = f(x)$

For an odd function $f(-x) = -f(x)$

$$\therefore \int_{-a}^a f(x) dx = \int_{-a}^0 f(x) dx + \int_0^a f(x) dx$$

$$= \int_0^a f(-x) dx + \int_0^a f(x) dx$$

$$= \int_0^a [f(-x) + f(x)] dx$$

$$= \int_0^a [f(x) + f(x)] dx, \text{ when } f \text{ is even fun}^n.$$

$$\int_0^a [-f(x) + f(x)] dx, \text{ when } f \text{ is odd fun}^n$$

$$= \begin{cases} 2 \int_0^a f(x) dx, & \text{when } f \text{ is even function.} \\ 0, & \text{when } f \text{ is odd function.} \end{cases}$$

$$\therefore \int_{-a}^a f(x) dx = \begin{cases} 2 \int_0^a f(x) dx, & \text{when } f \text{ is even.} \\ 0, & \text{when } f \text{ is odd.} \end{cases}$$

$$* I = \int_a^{2a} f(x) dx$$

$$\text{Put } 2a - x = t \Rightarrow x = 2a - t \Rightarrow dx = -dt$$

$$x \Rightarrow a, 2a \Rightarrow t \Rightarrow a, 0$$

$$\therefore I = \int_a^0 f(2a-t) (-dt) = \int_0^a f(2a-t) dt$$

$$= \int_0^a f(2a-x) dx$$

$$\therefore \int_0^{2a} f(x) dx = \int_0^a f(x) dx + \int_0^{2a} f(x) dx$$

$$= \int_0^a f(x) dx + \int_0^a f(2a-x) dx$$

$$= \int_0^a [f(x) + f(2a-x)] dx$$

$$= \begin{cases} 2 \int_0^a f(x) dx, & \text{if } f(2a-x) = f(x) \\ 0, & \text{if } f(2a-x) = -f(x) \end{cases}$$

$$\Rightarrow \int_0^{2a} f(x) dx = \begin{cases} 2 \int_0^a f(x) dx, & f(2a-x) = f(x) \\ 0, & f(2a-x) = -f(x) \end{cases}$$

* If $f(x)$ is a Periodic Function with Period T , then

$$\int_0^{nT} f(x) dx = n \int_0^T f(x) dx, \quad n \in \mathbb{N}.$$

Proof:- since $f(x)$ is a Periodic function with Period T , $f(x+T) = f(x)$

For $n \in \mathbb{N}$,

$$\begin{aligned} \int_0^{nT} f(x) dx &= \int_0^T f(x) dx + \int_T^{2T} f(x) dx + \dots + \int_{(n-1)T}^{nT} f(x) dx \\ &= \sum_{r=1}^n \int_{(r-1)T}^{rT} f(x) dx \end{aligned}$$

Put $x = y + T$
 Now, $\int_{(r-1)T}^{rT} f(x) dx = \int_{(r-1)T}^{rT} f(y) dy$

Put $x = y + (r-1)T \Rightarrow x \geq (r-1)T, rT$
 $\Rightarrow y = x - (r-1)T \quad y \geq 0, T$

$$\therefore \int_{(r-1)T}^{rT} f(x) dx = \int_0^T f[y + (r-1)T] dy = \int_0^T f(y) dy$$

as $f(x)$ is a Periodic function.

$$\int_0^{nT} f(x) dx = \sum_{r=1}^n \int_0^T f(x) dx$$

$$= n \int_0^T f(x) dx$$

$$\therefore \int_0^{nT} f(x) dx = n \int_0^T f(x) dx$$

eg: $I = \int_1^2 \frac{\sqrt{x} dx}{\sqrt{3-x} + \sqrt{x}}$

as we know, $\int_a^b f(x) dx = \int_a^b f(a+b-x) dx$

$$\int_1^2 \frac{\sqrt{x} dx}{\sqrt{3-x} + \sqrt{x}} = \int_1^2 \frac{\sqrt{1+2-x} dx}{\sqrt{3-(1+2-x)} + \sqrt{1+2-x}}$$

$$= \int_1^2 \frac{\sqrt{3-x} dx}{\sqrt{x} + \sqrt{3-x}}$$

$$\therefore I = \int_1^2 \frac{\sqrt{x} dx}{\sqrt{3-x} + \sqrt{x}} = \int_1^2 \frac{\sqrt{3-x} dx}{\sqrt{x} + \sqrt{3-x}}$$

$$\Rightarrow 2I = \int_1^2 \frac{\sqrt{x} dx}{\sqrt{3-x} + \sqrt{x}} + \int_1^2 \frac{\sqrt{3-x} dx}{\sqrt{x} + \sqrt{3-x}}$$

$$= \int_1^2 \left[\frac{\sqrt{x}}{\sqrt{3-x} + \sqrt{x}} + \frac{\sqrt{3-x}}{\sqrt{x} + \sqrt{3-x}} \right] dx$$

$$= \int_1^2 \left(\frac{\sqrt{x} + \sqrt{3-x}}{\sqrt{3-x} + \sqrt{x}} \right) dx$$

$$= \int_1^2 dx = [x]_1^2 = 2-1 = 1 \quad \therefore \boxed{I = 1/2}$$

$$* I = \int_0^{100} (\alpha - [\alpha]) d\alpha$$

$$\text{Let } f(\alpha) = \alpha - [\alpha]$$

$$\Rightarrow f(\alpha+1) = (\alpha+1) - [\alpha+1] = \alpha+1 - ([\alpha]+1)$$

$$\Rightarrow f(\alpha+1) = \alpha+1 - [\alpha] - 1 = \alpha - [\alpha]$$

$$\Rightarrow f(\alpha+1) = \alpha - [\alpha] = f(\alpha)$$

$\Rightarrow f(\alpha)$ is a periodic function with period 1.

$$\therefore \int_0^{100} (\alpha - [\alpha]) d\alpha = \int_0^{100} f(\alpha) d\alpha = 100 \int_0^1 (\alpha - [\alpha]) d\alpha$$

$$= 100 \int_0^1 \alpha d\alpha - 100 \int_0^1 [\alpha] d\alpha$$

$$= 100 \left[\frac{\alpha^2}{2} \right]_0^1 - 100 \times 0$$

$$= 100 \left(\frac{1}{2} - 0 \right) = 50$$

$$* I = \int_0^{\pi/4} \log(1 + \tan \theta) d\theta$$

$$= \int_0^{\pi/4} \log [1 + \tan (\theta + \frac{\pi}{4} - \theta)] d\theta \quad \left[\because \int_a^b f(x) dx = \int_a^b f(a+b-x) dx \right]$$

$$= \int_0^{\pi/4} \log [1 + \tan (\frac{\pi}{4} - \theta)] d\theta$$

$$= \int_0^{\pi/4} \log \left[1 + \frac{\tan \frac{\pi}{4} - \tan \theta}{1 + \tan \frac{\pi}{4} \cdot \tan \theta} \right] d\theta$$

$$= \int_0^{\pi/4} \log \left(1 + \frac{1 - \tan \theta}{1 + \tan \theta} \right) d\theta$$

$$= \int_0^{\pi/4} \log \left(\frac{2}{1 + \tan \theta} \right) d\theta$$

$$= \int_0^{\pi/4} [\log 2 - \log(1 + \tan \theta)] d\theta$$

$$= \int_0^{\pi/4} \log 2 d\theta - \int_0^{\pi/4} \log(1 + \tan \theta) d\theta$$

$$= \log 2 \int_0^{\pi/4} d\theta - I$$

$$\Rightarrow 2I = \log 2 [\theta]_0^{\pi/4} = \log 2 \left(\frac{\pi}{4}\right) = \frac{\pi}{4} \log 2$$

$$\Rightarrow \boxed{I = \frac{\pi}{8} \log 2}$$

$$\begin{aligned} * I &= \int_{-\pi/4}^{\pi/4} \cos^3 x dx = 2 \int_0^{\pi/4} \cos^3 x dx \quad (\because \cos^3 x \text{ is even.}) \\ &= 2 \int_0^{\pi/4} \cos^2 x \cdot \cos x dx \\ &= 2 \int_0^{\pi/4} (1 - \sin^2 x) \cos x dx \end{aligned}$$

Put $\sin x = z \Rightarrow dz = \cos x dx$

$x \Rightarrow 0, \pi/4 \Rightarrow z \Rightarrow \sin 0, \sin \pi/4 \Rightarrow z \Rightarrow 0, \frac{1}{\sqrt{2}}$

$$\therefore I = 2 \int_0^{1/\sqrt{2}} (1 - z^2) dz = 2 \left[z - \frac{z^3}{3} \right]_0^{1/\sqrt{2}}$$

$$= 2 \left(\frac{1}{\sqrt{2}} - \frac{\left(\frac{1}{\sqrt{2}}\right)^3}{3} \right) = 2 \left(\frac{1}{\sqrt{2}} - \frac{1}{6\sqrt{2}} \right) = \frac{2}{\sqrt{2}} \left(1 - \frac{1}{6} \right)$$

$$= \sqrt{2} \left(\frac{5}{6} \right)$$

$$* I = \int_{-\pi/6}^{\pi/6} \sin^3 x dx = 0 \quad (\because \sin^3 x \text{ is odd.})$$

$$* I = \int_0^{\pi} \sin^3 x \, dx$$

we know, $\sin^3(\pi - x) = \sin^3 x$

Then $2a = \pi \Rightarrow a = \pi/2$

Thus, $\int_0^{2a} f(x) \, dx = 2 \int_0^a f(x) \, dx$

$$\begin{aligned} \Rightarrow I &= \int_0^{\pi} \sin^3 x \, dx = 2 \int_0^{\pi/2} \sin^3 x \, dx \\ &= 2 \int_0^{\pi/2} (1 - \cos^2 x) \sin x \, dx \end{aligned}$$

Put $z = \cos x \Rightarrow dz = -\sin x \, dx$

$x \rightarrow 0, \pi/2 \Rightarrow z \rightarrow \cos 0, \cos \pi/2 \Rightarrow z \rightarrow 1, 0$

$$\begin{aligned} \therefore I &= 2 \int_1^0 (1 - z^2)(-dz) = 2 \int_0^1 (1 - z^2) dz = 2 \left[z - \frac{z^3}{3} \right]_0^1 \\ &= 2 \left(1 - \frac{1}{3} \right) = \frac{2}{3} \times 2 = \frac{4}{3} \end{aligned}$$

$$* I = \int_0^{\pi/2} \ln \sin x \, dx = \int_0^{\pi/2} \ln \sin(\pi/2 - x) \, dx$$

$$\Rightarrow I = \int_0^{\pi/2} \ln \cos x \, dx$$

$$\therefore 2I = \int_0^{\pi/2} \ln \sin x \, dx + \int_0^{\pi/2} \ln \cos x \, dx$$

$$= \int_0^{\pi/2} (\ln \sin x + \ln \cos x) \, dx$$

$$= \int_0^{\pi/2} \ln(\sin x \cdot \cos x) \, dx$$

$$= \int_0^{\pi/2} \ln \left(\frac{2 \sin x \cdot \cos x}{2} \right) dx = \frac{1}{2} \int_0^{\pi/2} \ln \left(\frac{\sin 2x}{2} \right) dx$$

$$= \int_0^{\pi/2} \ln(\sin 2x) dx - \int_0^{\pi/2} \ln 2 dx$$

Now, $\int_0^{\pi/2} \ln(\sin 2x) dx = \int_0^{\pi} \ln \sin z \frac{dz}{2}$

[\because Put $2x = z \Rightarrow dz = 2dx \Rightarrow dx = \frac{dz}{2}$

$x \rightarrow 0, \frac{\pi}{2} \Rightarrow z \Rightarrow \frac{2 \cdot 0}{2}, 2 \cdot \frac{\pi}{2} \Rightarrow z \rightarrow 0, \pi$]

$$\therefore \int_0^{\pi/2} \ln \sin 2x dx = \frac{1}{2} \int_0^{\pi} \ln \sin z dz$$

$$= \frac{1}{2} \int_0^{2 \cdot \pi/2} \ln \sin z dz$$

Now $\sin(2 \cdot \frac{\pi}{2} - z) = \sin(\pi - z) = \sin z$

$$\therefore \frac{1}{2} \int_0^{2 \cdot \pi/2} \ln \sin z dz = \frac{1}{2} \cdot 2 \int_0^{\pi/2} \ln \sin z dz = \int_0^{\pi/2} \ln \sin z dz$$

$$= I$$

Thus, $\int_0^{\pi/2} \ln \sin z dz - \ln 2 \int_0^{\pi/2} dz = 2I$

$$\Rightarrow I - \ln 2 [z]_0^{\pi/2} = 2I$$

$$\Rightarrow I = -\ln 2 \left(\frac{\pi}{2} - 0 \right) \Rightarrow \boxed{I = -\frac{\pi}{2} \ln 2 = \frac{\pi}{2} \ln \left(\frac{1}{2} \right)}$$

Application of Integration

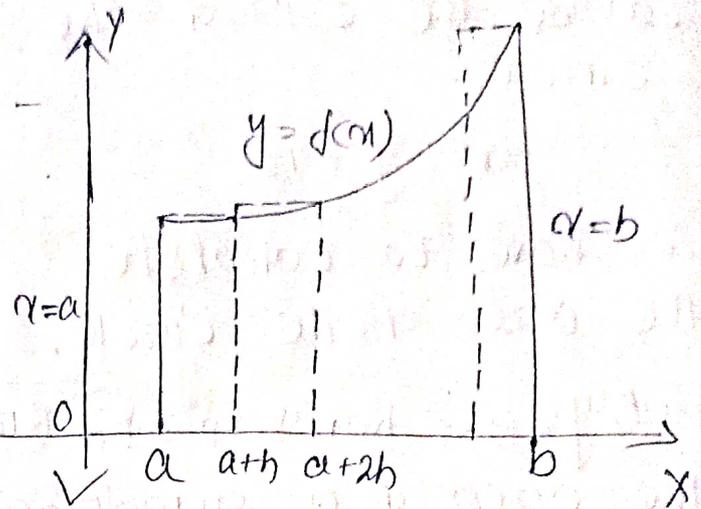
Area enclosed by a curve and x-axis :-

Consider a function

$$y = f(x)$$

and the ordinates

$$x = a \text{ and } x = b.$$



Then, the definite integral

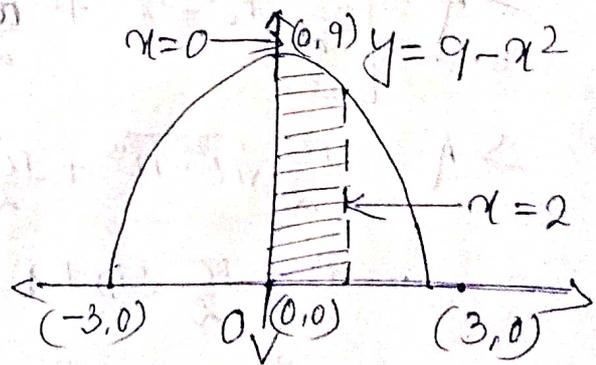
$$A = \int_a^b f(x) dx$$

represents the area under the curve $y=f(x)$ and the x-axis with the ordinates

$$x = a \text{ and } x = b.$$

Ex:- Area of the region enclosed by $y=9-x^2$ and $y=0$ and the ordinates $x=0$ and $x=2$ is given by -

$$\begin{aligned} A &= \int_a^b f(x) dx \\ &= \int_0^2 (9-x^2) dx \end{aligned}$$



$$\begin{aligned} &= \left[9x - \frac{x^3}{3} \right]_0^2 \\ &= 18 - \frac{8}{3} = \frac{54-8}{3} = \frac{46}{3} \text{ sq. units} \end{aligned}$$

Hence the area enclosed by the curve $y=9-x^2$ and $y=0$ and the ordinates $x=0$ and $x=2$ is $\frac{46}{3}$ sq. units.

Area of the circle with centre at origin:-

consider the eqn of the circle -

$$x^2 + y^2 = r^2$$

we have to calculate the area of the circle.

firstly, we have to find $(-r, 0)$ the area of a quadrant of the circle $x^2 + y^2 = r^2$.

$$\text{Then } y^2 = r^2 - x^2$$

$$\Rightarrow y = \sqrt{r^2 - x^2} = f(x)$$

Hence the area of the quadrant of the circle is -

$$A = \int_0^r f(x) dx = \int_0^r \sqrt{r^2 - x^2} dx$$

$$\Rightarrow A = \left[\frac{x}{2} \sqrt{r^2 - x^2} + \frac{r^2}{2} \sin^{-1} \left(\frac{x}{r} \right) \right]_0^r$$

$$\Rightarrow A = \frac{r}{2} \sqrt{r^2 - r^2} + \frac{r^2}{2} \sin^{-1} \left(\frac{r}{r} \right) - 0 - \frac{r^2}{2} \sin^{-1} \left(\frac{0}{r} \right)$$
$$= 0 + \frac{r^2}{2} \sin^{-1} 1 - \frac{r^2}{2} \sin^{-1}(0)$$

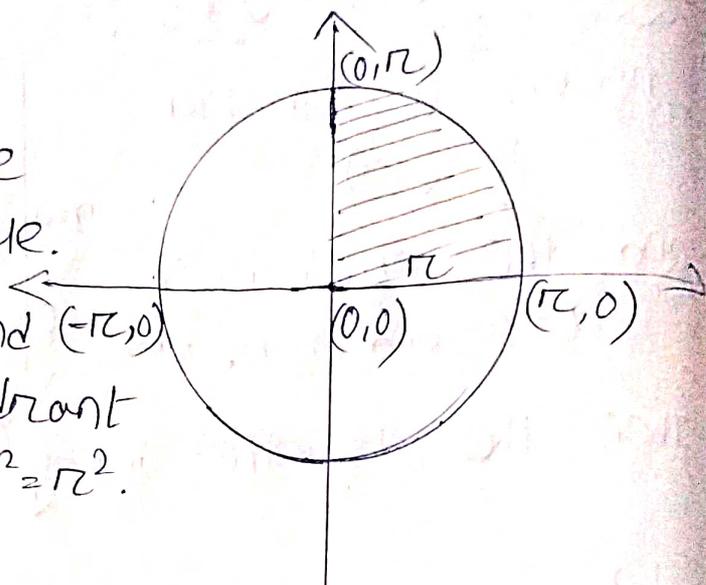
$$= \frac{r^2}{2} \cdot \frac{\pi}{2} - \frac{r^2}{2} \cdot 0 = \frac{1}{4} \pi r^2 \Rightarrow \boxed{A = \frac{1}{4} \pi r^2}$$

Hence the area of the quadrant of the circle is $\frac{1}{4} \pi r^2$ sq. units.

Thus the area of the circle = $4 \times A$

$$= 4 \times \frac{1}{4} \pi r^2 = \pi r^2$$

Hence area of circle is πr^2 sq. units.



For the semicircle, Area of the semicircle.

$$A = \int_{-r}^r f(x) dx = \int_{-r}^r \sqrt{r^2 - x^2} dx$$

$$f(x) = \sqrt{r^2 - x^2} \Rightarrow f(-x) = \sqrt{r^2 - (-x)^2} = \sqrt{r^2 - x^2}$$

$$\Rightarrow f(x) = f(-x) \Rightarrow f(x) \text{ is even function.}$$

$$A = \int_{-r}^r f(x) dx = 2 \int_0^r f(x) dx$$

$$\Rightarrow A = 2 \int_0^r \sqrt{r^2 - x^2} dx$$

$$= 2 \times \frac{1}{4} \pi r^2$$

$$= \frac{1}{2} \pi r^2$$

$$\Rightarrow \boxed{A = \frac{1}{2} \pi r^2}$$