LEARNING MATERIAL
ON
VECTOR ALGEBRA

SEMESTER : II
DEPARTMENT : MATHEMATICS AND SCIENCE
SUBJECT NAME : ENGINEERING MATHEMATICS-II
SUBJECT CODE : TH.3

PREPARED BY
DEEPAK KUMAR SAHOO
LECTURER IN MATHEMATICS

DEPARTMENT OF MATHEMATICS & SCIENCE,
ORISSA SCHOOL OF MINING ENGINEERING,
KEONJHAR
758001
Website: www.osme.co.in
Email id: osmemath.science@gmail.com

Vectors

There are two types of physical quantities

(i) Scalar (having magnitude only)

(ii) Vector (having both magnitude and direction).

Representation of a vector

In general a vector is represented by small or capital letter and an arrow over it.

e.g. $\vec{A}$, $\vec{a}$, $\overrightarrow{PQ}$.

If $\vec{v} = \overrightarrow{PQ}$, P is the initial point and Q is the terminal point.

Magnitude of a vector

If $\vec{v} = \overrightarrow{PQ}$, $|\vec{v}| = PQ$ is called magnitude of the vector $\vec{v}$.

Direction of $\vec{v}$ is from P to Q.

Types of vectors

Unit vector – If $|\vec{v}|=1$ then $\vec{v}$ is called a unit vector and is usually denoted by $\hat{v}$.

$$\hat{v} = \frac{\vec{v}}{|\vec{v}|}.$$

\# Unit vector along X – axis $\rightarrow \hat{i}$

Y – axis $\rightarrow \hat{j}$

Z – axis $\rightarrow \hat{k}$

Zero Vector – (Null vector)

If $|\vec{v}| = 0$, then $\vec{v}$ is called zero vector or null vector.

It is denoted by $\vec{0}$.

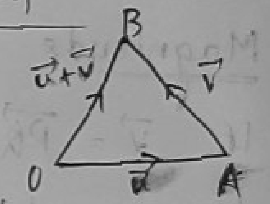
It has infinitely many directions.

Equal Vectors

Two vectors are said to be equal if they have same magnitude and same direction.

Addition of two vectors

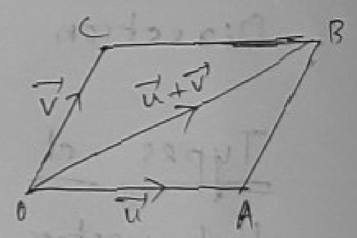
(i) Triangle's law of vector addition



If $\vec{u} = \vec{OA}$, $\vec{v} = \vec{AB}$

then $\vec{u} + \vec{v} = \vec{OA} + \vec{AB} = \vec{OB}$.

(ii) Parallelogram law of vector addition.



If $\vec{u} = \vec{OA}$, $\vec{v} = \vec{OC}$

then $\vec{u} + \vec{v} = \vec{OA} + \vec{OC} = \vec{OB}$

Position Vector

Let O be a fixed point.

Then the vector $\vec{OP}$ is called the position vector of point P relative to O and is denoted by $\vec{p} = \vec{OP}$.

\# Position vector of any point $P(x, y, z)$ is given by
$= x\hat{i} + y\hat{j} + z\hat{k}$.

Position vector of $\vec{AB}$ = P.V. of $\vec{B}$ – P.V. of $\vec{A}$
$= \vec{OB} - \vec{OA}$.
$= \vec{b} - \vec{a}$

Note -

1. Magnitude of a vector $\vec{v} = x\hat{i} + y\hat{j} + z\hat{k}$ is $|\vec{v}| = \sqrt{x^2 + y^2 + z^2}$.

i.e. $\boxed{\text{If } \vec{v} = x\hat{i} + y\hat{j} + z\hat{k}, \text{ then } |\vec{v}| = \sqrt{x^2+y^2+z^2}}$

2. If $\vec{u} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$, $\vec{v} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$

then $\vec{u} + \vec{v} = (x_1 + x_2)\hat{i} + (y_1 + y_2)\hat{j} + (z_1 + z_2)\hat{k}$

3. If $\vec{A} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$, $\vec{B} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$

then $\overrightarrow{AB} = \vec{B} - \vec{A} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$.

$$|\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Properties -

Let $\vec{u}, \vec{v}, \vec{w}$ are three vectors and α, β are two scalars.

1. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
2. $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$
3. $\alpha(\vec{u} + \vec{v}) = \alpha\vec{u} + \alpha\vec{v}$
4. $(\alpha + \beta)\vec{u} = \alpha\vec{u} + \beta\vec{u}$
5. $(\alpha\beta)\vec{u} = \alpha(\beta\vec{u}) = \beta(\alpha\vec{u})$
6. $1.\vec{u} = \vec{u}$
7. $0.\vec{u} = \vec{0}$
8. $\alpha.\vec{0} = \vec{0}$

Parallel and co-linear vectors

Two vectors are parallel if one is a scalar multiple of other.

e.g. $\vec{u}$ and $\vec{v}$ are parallel iff $\vec{u} = k\vec{v}$

Two vectors are co-linear if they are parallel.

Note -

1. Two vectors $\vec{u} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$, $\vec{v} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$ are parallel if $\boxed{\frac{x_1}{x_2} = \frac{y_1}{y_2} = \frac{z_1}{z_2}}$

Section Formula

Let P and Q be two points with position vectors $\vec{p}$ and $\vec{q}$ and R be a point on line segment $\overline{PQ}$ which divides it in ratio $m:n$

If $\vec{r}$ is the position vector of point R,

then $\boxed{\vec{r} = \frac{m\vec{q} + n\vec{p}}{m+n}}$

Note: If $R(\vec{r})$ is the midpoint of $\overline{PQ}$

then $\vec{r} = \frac{\vec{p} + \vec{q}}{2}$

Note - If $\vec{v} = x\hat{i} + y\hat{j} + z\hat{k}$ or (x, y, z)

then (i) x, y, z are scalar components of $\vec{v}$.

(ii) $x\hat{i}, y\hat{j}, z\hat{k}$ are vector components of $\vec{v}$

Scalar Product or Dot product

1. Let $\vec{a}$ and $\vec{b}$ are two vectors and θ be the angle between them, then there dot product is denoted by $\vec{a}.\vec{b}$ defined by $\boxed{\vec{a}.\vec{b} = |\vec{a}|.|\vec{b}| \cos\theta}$

2. If $\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$
   $\vec{b} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$
   then $\boxed{\vec{a}.\vec{b} = x_1x_2 + y_1y_2 + z_1z_2}$

3. Angle between $\vec{a}$ and $\vec{b}$
   $$\boxed{\theta = \cos^{-1}\frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}}$$

4. If two vectors $\vec{a}$ and $\vec{b}$ are perpendicular then their dot product is zero.
   i.e. $\boxed{\text{If } \vec{a} \perp \vec{b} \text{ then } \vec{a}.\vec{b} = 0}$

5. $\vec{a}.\vec{a} = |\vec{a}|^2$.
   $\hat{i}.\hat{i} = \hat{j}.\hat{j} = \hat{k}.\hat{k} = 1$

6. $\hat{i}.\hat{j} = \hat{j}.\hat{k} = \hat{k}.\hat{i} = 0$

7. $\vec{a}.\vec{b} = \vec{b}.\vec{a}$.

8. $\vec{a}.(\vec{b}+\vec{c}) = \vec{a}.\vec{b} + \vec{a}.\vec{c}$

Geometrical meaning of dot product

The dot product of two vectors is equal to magnitude of one vector multiplied by the projection of the other vector on it.

i.e. $\vec{a}\cdot\vec{b} = |\vec{a}|\cdot|\vec{b}| \cos\theta$

$$= |\vec{a}|\,(|\vec{b}| \cos\theta)$$

$$= |\vec{a}|\,(\text{scalar projection of } \vec{b} \text{ on } \vec{a})$$

Note: (i) scalar projection of $\vec{b}$ on $\vec{a}$

$$= |\vec{b}| \cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$$

(ii) scalar projection of $\vec{a}$ on $\vec{b}$

$$= |\vec{a}| \cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$$

(iii) vector projection of $\vec{b}$ on $\vec{a}$

$$= \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\right)\vec{a}$$

(iv) vector projection of $\vec{a}$ on $\vec{b}$

$$= \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\right)\vec{b}$$

Note

If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

then, $\boxed{\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3}$

Now, $|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

$|\vec{b}| = \sqrt{b_1^2 + b_2^2 + b_3^2}$

$$\boxed{\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|} = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\cdot\sqrt{b_1^2 + b_2^2 + b_3^2}}}$$

$$\Rightarrow \theta = \cos^{-1}\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|}$$

$$= \cos^{-1}\left(\frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\,\sqrt{b_1^2 + b_2^2 + b_3^2}}\right)$$

Cross Product or Vector Product

1. Let $\vec{a}$ and $\vec{b}$ are two vectors and θ be the angle between them, then their cross product is denoted by $\vec{a} \times \vec{b}$

defined by $\boxed{\vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}| \sin\theta \ \hat{n}}$

where $\hat{n}$ is the unit vector perpendicular to both $\vec{a}$ and $\vec{b}$.

2. Direction of $\vec{a} \times \vec{b} = \boxed{\hat{n} = \dfrac{\vec{a} \times \vec{b}}{|\vec{a}|\,|\vec{b}| \sin\theta}}$

3. If two vectors are parallel then their cross product is zero vector.

$\boxed{\text{If } \vec{a} \parallel \vec{b} \text{, then } \vec{a} \times \vec{b} = \vec{0}}$

4. $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

5. $\alpha(\vec{a} \times \vec{b}) = (\alpha \vec{a} \times \vec{b}) = \vec{a} \times (\alpha \vec{b})$

6. $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

$(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$

7. $\vec{a} \times \vec{a} = \vec{0}$

$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \vec{0}$.

8. $\hat{i} \times \hat{j} = \hat{k}$

$\hat{j} \times \hat{k} = \hat{i}$

$\hat{k} \times \hat{i} = \hat{j}$

Angle Between $\vec{a} \times$ and $\vec{b}$

$$\vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}| \sin\theta \cdot \hat{n}$$

$$\Rightarrow |\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}| \sin\theta$$

$$\Rightarrow \boxed{\sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}|\,|\vec{b}|}}$$

$$\Rightarrow \boxed{\theta = \sin^{-1} \frac{|\vec{a} \times \vec{b}|}{|\vec{a}|\,|\vec{b}|}}$$

Note - If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$.

then

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= (a_2b_3 - a_3b_2)\hat{i} - (a_1b_3 - a_3b_1)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}$$

Geometrical meaning of cross product

1. Let $\vec{a}$ and $\vec{b}$ are two sides of a triangle.

   Area of triangle $= \frac{1}{2}|\vec{a} \times \vec{b}|$.

2. Let $\vec{a}$ and $\vec{b}$ are two sides of a parallelogram

   Area of parallelogram $= |\vec{a} \times \vec{b}|$.

3. If $\vec{d_1}$ and $\vec{d_2}$ are diagonals of a parallelogram

   Area of parallelogram $= \frac{1}{2}|\vec{d_1} \times \vec{d_2}|$

Solved Examples

1. If position vectors of two given points A and B are $\hat{i}-2\hat{j}+3\hat{k}$ and $2\hat{i}-\hat{j}+\hat{k}$ respectively, find magnitude and direction $\overrightarrow{AB}$.

Solution: P.V. of A $= \hat{i}-2\hat{j}+3\hat{k}$

P.V. of B $= 2\hat{i}-\hat{j}+\hat{k}$

$\overrightarrow{AB}$ = P.V. of B − P.V. of A

$$= 2\hat{i}-\hat{j}+\hat{k} - (\hat{i}-2\hat{j}+3\hat{k})$$

$$= \hat{i}+\hat{j}-2\hat{k}$$

Magnitude of $\overrightarrow{AB} = |\overrightarrow{AB}| = \sqrt{1^2+1^2+(-2)^2}$

$$= \sqrt{1+1+4} = \sqrt{6}$$

Direction of $\overrightarrow{AB} = \hat{AB}$

$$= \frac{\overrightarrow{AB}}{|\overrightarrow{AB}|} = \frac{\hat{i}+\hat{j}-2\hat{k}}{\sqrt{6}}$$

2. If two vectors $\vec{a} = \hat{i}-2\hat{j}+p\hat{k}$ and $\vec{b} = \hat{i}+q\hat{j}+\hat{k}$ are equal find p and q.

Solution. $\vec{a} = \vec{b}$

$\Rightarrow \hat{i}-2\hat{j}+p\hat{k} = \hat{i}+q\hat{j}+\hat{k}$

$\Rightarrow p=1, q=-2$ (comparing components of $\hat{i}, \hat{j}$ and $\hat{k}$)

$\therefore p=1$

$q=-2$

3. If $\vec{a} = 2\hat{i} + 3\hat{j} - 4\hat{k}$, $\vec{b} = \hat{i} - 2\hat{j} + \hat{k}$, $\vec{c} = -\hat{i} + 2\hat{j} - \hat{k}$

find $\vec{a} + 2\vec{b} - 3\vec{c}$

Solution:

$$\vec{a} + 2\vec{b} - 3\vec{c}$$

$$= (2\hat{i} + 3\hat{j} - 4\hat{k}) + 2(\hat{i} - 2\hat{j} + \hat{k}) - 3(-\hat{i} + 2\hat{j} - \hat{k})$$

$$= 2\hat{i} + 3\hat{j} - 4\hat{k} + 2\hat{i} - 4\hat{j} + 2\hat{k} + 3\hat{i} - 6\hat{j} + 3\hat{k}$$

$$= 7\hat{i} - 7\hat{j} + \hat{k}.$$

4. The P.V. of three points A, B and C are $2\hat{i} + \hat{j} - \hat{k}$, $\hat{i} - 2\hat{j} + \hat{k}$ and $3\hat{i} + 4\hat{j} - 3\hat{k}$, respectively. Show that A, B and C are co-linear.

Solution.

$$\vec{AB} = \hat{i} - 2\hat{j} + \hat{k} - (2\hat{i} + \hat{j} - \hat{k}) = -\hat{i} - 3\hat{j} + 2\hat{k}$$

$$\vec{BC} = 3\hat{i} + 4\hat{j} - 3\hat{k} - (\hat{i} - 2\hat{j} + \hat{k}) = 2\hat{i} + 6\hat{j} - 4\hat{k}$$

$$= -2(\vec{AB})$$

$\Rightarrow \vec{BC} = -2 \cdot \vec{AB}$

$\Rightarrow \vec{BC} \parallel \vec{AB}$

$\Rightarrow$ A, B and C are co-linear

5. If $\vec{a}$ and $\vec{b}$ are parallel vectors where $\vec{a} = \hat{i} + 2\hat{j} - d_1\hat{k}$, $\vec{b} = d_2\hat{i} + 2\hat{j} + \hat{k}$, find d_1 and d_2

Solution:

$$\vec{a} \parallel \vec{b}$$

$$\therefore \frac{1}{d_2} = \frac{2}{2} = \frac{-d_1}{1}$$

$\Rightarrow d_2 = 1$, $d_1 = -1$ (By formula for parallel vectors)

6. If $\vec{a} = 2\hat{i} + \hat{j} - \hat{k}$, $\vec{b} = \hat{i} - 2\hat{j} + 3\hat{k}$, find $\vec{a}\cdot\vec{b}$!

Soln, $\vec{a}\cdot\vec{b} = 2.1 + 1.(-2) + (-1).3$ (see formula for $\vec{a}\cdot\vec{b}$)

$= 2 - 2 - 3 = -3$.

7. Find x if the vectors $\vec{a} = x\hat{i} - \hat{j} + 2\hat{k}$ and $\vec{b} = \hat{i} + \hat{j} - \hat{k}$ are perpendicular.

Soln / Vectors $\vec{a}$ and $\vec{b}$ are pendicular iff

$\vec{a}\cdot\vec{b} = 0$

$\Rightarrow (x\hat{i} - \hat{j} + 2\hat{k}).(\hat{i} + \hat{j} - \hat{k}) = 0$

$\Rightarrow x.1 + (-1).1 + 2.(-1) = 0$

$\Rightarrow x - 1 - 2 = 0$

$\Rightarrow x = 3$

8. Find angle between the vectors $\vec{a} = 2\hat{i} - 2\hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + \hat{j} + 2\hat{k}$.

Soln, Let θ be the angle between $\vec{a}$ and $\vec{b}$.

Then, $\theta = \cos^{-1}\left(\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|}\right)$

Now, $|\vec{b}| = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{1+1+4} = \sqrt{6}$.

$|\vec{a}| = \sqrt{2^2 + (-2)^2 + 1^2} = \sqrt{4+4+1} = 3$.

$\vec{a}\cdot\vec{b} = (2\hat{i} - 2\hat{j} + \hat{k}).(\hat{i} + \hat{j} + 2\hat{k}) = 2.1 - 2.1 + 1.2 = 2$

Now, $\theta = \cos^{-1}\left(\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|}\right) = \cos^{-1}\left(\dfrac{2}{\sqrt{6}.3}\right)$

$= \cos^{-1}\left(\dfrac{\sqrt{2}}{3\sqrt{3}}\right)$

9. Find scalar and vector projections of the vector $2\hat{i}-\hat{j}+\hat{k}$ on the vector $\hat{i}+\hat{j}-3\hat{k}$.

Soln. let $\vec{a} = 2\hat{i}-\hat{j}+\hat{k}$, $\vec{b} = \hat{i}+\hat{j}-3\hat{k}$

$|\vec{a}| = \sqrt{2^2+(-1)^2+1^2} = \sqrt{6}$

$|\vec{b}| = \sqrt{1^2+1^2+(-3)^2} = \sqrt{11}$

$\vec{a}\cdot\vec{b} = (2\hat{i}-\hat{j}+\hat{k})\cdot(\hat{i}+\hat{j}-3\hat{k})$

$= 2\cdot1+(-1)\cdot1+1\cdot(-3)$

$= 2-1-3 = -2$

Now, scalar projection of $\vec{a}$ on $\vec{b}$

$$= \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}\right) = \frac{-2}{\sqrt{11}}$$

vector projection of $\vec{a}$ on $\vec{b}$

$$= \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\right)\vec{b} = \frac{-2}{11}\cdot(\hat{i}+\hat{j}-3\hat{k})$$

10. If $\vec{a} = 2\hat{i}-\hat{j}+\hat{k}$, $\vec{b} = 3\hat{i}+4\hat{j}-\hat{k}$ find $\vec{a}\times\vec{b}$.

Soln. $\vec{a} = 2\hat{i}-\hat{j}+\hat{k}$

$\vec{b} = 3\hat{i}+4\hat{j}-\hat{k}$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ 3 & 4 & -1 \end{vmatrix}$$

$= \hat{i}(1-4) - \hat{j}(-2-3) + \hat{k}(8-(-3))$

$= -3\hat{i}+5\hat{j}+11\hat{k}$.

11. Find a unit vector perpendicular to the vectors $2\hat{i}-3\hat{j}+\hat{k}$ and $-\hat{i}+2\hat{j}-\hat{k}$.

Sol^n. Let $\vec{a} = 2\hat{i}-3\hat{j}+\hat{k}$, $\vec{b} = -\hat{i}+2\hat{j}-\hat{k}$

Vector perpendicular to both $\vec{a}$ and $\vec{b} = \vec{a}\times\vec{b}$.

Unit vector perpendicular to $\vec{a}$ and $\vec{b} = \dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$.

Now, $$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -3 & 1 \\ -1 & 2 & -1 \end{vmatrix}$$

$$= \hat{i}(3-2) +- \hat{j}(-2+1) + \hat{k}(4-3)$$

$$= \hat{i}+\hat{j}+\hat{k}.$$

$$|\vec{a}\times\vec{b}| = \sqrt{1^2+1^2+1^2} = \sqrt{3}.$$

Now, $$\frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|} = \frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}.$$

$\therefore$ Unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ is $\dfrac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}$.

12. If the area of the parallelogram whose sides are the vectors $\hat{i}-\hat{j}+\hat{k}$ and $2\hat{i}+\hat{j}-\hat{k}$.

Sol^n, let $\vec{a} = \hat{i}-\hat{j}+\hat{k}$, $\vec{b} = 2\hat{i}+\hat{j}-\hat{k}$.

Area of the parallelogram whose sides are $\vec{a}$ and $\vec{b} = |\vec{a}\times\vec{b}|$

Now, $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{vmatrix}$

$= \hat{i}(1-1) - \hat{j}(-1-2) + \hat{k}(1-(-2))$

$= 3\hat{j} + 3\hat{k}$

$\Rightarrow |\vec{a} \times \vec{b}| = \sqrt{3^2 + 3^2} = 3\sqrt{2}$

$\therefore$ Area of the parallelogram $= 3\sqrt{2}$ sq. units.

13. Find area of the triangle whose vertices are $A(1,2,3)$, $B(-1,-1,0)$, $C(1,-1,0)$.

Soln. $\vec{AB}$ = P.V. of B − P.V. of A $= (-1-1)\hat{i} + (-1-2)\hat{j} + (0-3)\hat{k}$

$= -2\hat{i} - 3\hat{j} - 3\hat{k}$

$\vec{BC}$ = P.V of C − P.V. of B $= (1-(-1))\hat{i} + (-1+1)\hat{j} + (0-0)\hat{k}$

$= 2\hat{i}$.

Area of $\Delta ABC = \frac{1}{2}|\vec{AB} \times \vec{BC}|$

$\vec{AB} \times \vec{BC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & -3 & -3 \\ 2 & 0 & 0 \end{vmatrix}$

$= 0.\hat{i} - 6\hat{j} + 6\hat{k} = -6\hat{j} + 6\hat{k}$

$|\vec{AB} \times \vec{BC}| = \sqrt{(-6)^2 + 6^2} = 6\sqrt{2}$

Area of $\Delta ABC = \frac{1}{2}|\vec{AB} \times \vec{BC}| = \frac{1}{2} \times 6\sqrt{2} = 3\sqrt{2}$ sq unit

14. Find the area of the parallelogram whose diagonals are the vectors $3\hat{i}+\hat{j}-2\hat{k}$ and $\hat{i}-3\hat{j}+4\hat{k}$

Soln, Let $\vec{d_1} = 3\hat{i}+\hat{j}-2\hat{k}$, $\vec{d_2} = \hat{i}-3\hat{j}+4\hat{k}$.

Area of the parallelogram whose diagonals are the vectors d_1 and d_2 $= \frac{1}{2}|\vec{d_1} \times \vec{d_2}|$

Now, $$\vec{d_1} \times \vec{d_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & -2 \\ 1 & -3 & 4 \end{vmatrix}$$

$$\cancel{= \hat{i}(12+2) - \hat{j}\,(}$$

$$= \hat{i}(4-6) - \hat{j}(12+2) + \hat{k}(-9-1)$$

$$= -2\hat{i} - 14\hat{j} - 10\hat{k}.$$

Now, $$|\vec{d_1} \times \vec{d_2}| = \sqrt{(-2)^2 + (-14)^2 + (-10)^2}$$

$$= \sqrt{4+196+100} = \sqrt{300} = 10\sqrt{3}.$$

$\therefore$ Area of the parallelogram

$$= \frac{1}{2}|\vec{d_1} \times \vec{d_2}|$$

$$= \frac{1}{2} \times 10\sqrt{3}$$

$$= 5\sqrt{3} \text{ sq units}.$$